"ASYMPTOTIC BEHAVIOR OF THE AVERAGE ADJACENCIES FOR SKELETON-REGULAR TRIANGULAR AND TETRAHEDRAL PARTITIONS"

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Asymptotic behavior of the average adjacencies for skeleton-regular triangular and tetrahedral partitions

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Abstract

In this paper a general class of simplex partitions that we call *skeleton-regular* partitions are studied. For any conforming mesh, the individual application of a skeleton-regular partition produces a conforming mesh such that all the topological elements of the same dimension are subdivided in the same number of son-elements. Every skeleton-regular partition has associated special constitutive (recurrence) equations. We show that the asymptotic number of average adjacencies of the topological elements defining the mesh is finite, when the partition is iteratively and globally applied over any initial conforming mesh. We prove that this limit is identical for any skeleton-regular partition in 2D. On the contrary, in three dimensions different values are obtained depending on the considered partition. We study the average adjacencies associated with the most common skeleton-regular partitions in 3D, and they are compared with the values reported for actual tetrahedral meshes for finite element calculations.

Key words: partitions, adjacencies, triangular and tetrahedral meshes.

AMS subject classifications: 51M20, 52B10.

1   Introduction

The *kissing number* of a convex body K is the maximum number of congruent copies of K that can touch K without overlapping with each other. For instance, the kissing number of the 2D ball B^2 is 6. The question about the number for the 3D ball B^3 caused a dispute between Isaac Newton and David Gregory. Newton conjectured that the answered was 12 while Gregory thought 13 was possible. It took 180 years before the question was answered: Hoppe [1] proved that Newton was

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right [2]. Perhaps for this reason the kissing number of a convex body K is also known as the *Newton number* of K.

On the other hand, in the area of numerical methods a considerable effort has been done for designing and implementing a suitable distribution of points or elements into the problem domain. Very often we are interested in measuring the goodness of the partition (or the triangulation). In this sense several smoothing or improvement techniques have been developed, so a better triangulation (in the simplicial case) can be achieved (see, for example references [3, 4]). Some topological and geometrical regularity measures for simplices (triangles in 2D, tetrahedra in 3D) and for the whole simplicial grid (*triangulation*) have been proposed in literature [5, 6]. For example, Kenji Shimada [6] proposed in his Thesis the following one, for a triangulation τ:

$$(1) \qquad \epsilon_\tau = \frac{1}{n} \sum_{i=1}^n |\delta_i - D|,$$

where $D = 6$ for triangles, and $D = 12$ for tetrahedra, and δ_i represents the *degree* of node i, that is the number of nodes connected to the i-th interior node, and n is the total number of interior nodes in the domain. Thus, in general, as elements become more equilateral, the mesh irregularity approaches 0, but vanishes only when all the nodes have D neighbors, a rare situation. Otherwise, it has a positive value that designates how much the mesh differs from a perfectly regular triangular lattice. Recently Buss and Simpson have proved that planar mesh refinement cannot be both local and regular [7], and in higher dimensions it is well known that there is not a regular simplicial partition of the space.

In this paper we investigate the asymptotic behavior of the average of the degree of the nodes when the number of global refinements tends to infinite, that is the asymptotic average of the degree of the nodes. Moreover, all possible averages of the adjacencies of the topological elements are studied for a class of simplex partitions. These special partitions that we call *skeleton-regular* partitions are characterized because they subdivide all the topological elements of the same dimension (the elements of the k-skeleton) of the mesh in the same number of topological *son-elements*, resulting besides a conforming triangulation.

The paper is organized as follows. In the next section, we introduce some definitions and notations. The third section is for explaining some of the most common simplex partions used in practice. Finally the asymptotic behavior of the average of the adjacencies of the topological elements is presented for regular partitions in 2 and 3 dimensions.

2 Definitions and notations

Some elementary definitions and notations are summarized here.

Definition 2.1 (simplices) A closed subset $T \in \mathbb{R}^n$ is called a *(k)-simplex*, $0 \leq k \leq n$ if T is the convex linear hull of $k+1$ vertices $x^0, x^1, \ldots, x^k \in \mathbb{R}^n$:

$$(2) \qquad T = [x^{(0)}, x^{(1)}, \ldots, x^{(k)}] := \left\{ \sum_{j=0}^{k} \lambda_j x^{(j)} \mid \sum_{j=0}^{k} \lambda_j = 1; \ \lambda_j \in [0,1], \ 0 \leq j \leq k \right\}.$$

■

The vertex ordering is important in some partitions (cfr. [11]). Other times the vertex numbering is different in two simplices T and T' but still T and T' denote the same subset of $\mathbb{R}^n$; it is said that T coincides with T' in the sense of sets. We are not going to use the vertex ordering in this study.

If $k = n$ the T is simply called *simplex* or *triangle* of $\mathbb{R}^n$. (2)- and (3)-simplices are called triangles and tetrahedra as usual. In the following the simplices, or $n-$simplices will be called *triangles*, because we are going to restrict our interest to two and three dimensions. So, from now on a *triangle* will be either a 2D triangle or a 3D triangle (a tetrahedron).

Definition 2.2 (triangulation) Let Ω be any bounded set in $\mathbb{R}^2$, or $\mathbb{R}^3$ with no-empty interior, and polygonal boundary $\partial\Omega$, and consider a partition of Ω into a set of triangles $\tau = \{t_1, t_2, t_3, \ldots, t_n\}$. Then we say that τ is a triangulation if the following properties hold:

1. $\Omega = \bigcup t_i$
2. $interior(t_i) \neq \emptyset, \ \forall t_i \in \Omega$
3. $interior(t_i) \bigcap interior(t_j)$, if $i \neq j$ ■

Definition 2.3 (conforming triangulation) A triangulation τ of a bounded set Ω is called conforming (some authors prefer consistent, or compatible) if any pair of adjacent simplices share either an entire face or edge, or a common vertex. ■

Definition 2.4 (k-face) Let $T = [x^{(0)}, x^{(1)}, \ldots, x^{(n)}]$ be a (n)-simplex in $\mathbb{R}^n$. A k-simplex $S = [y^{(0)}, y^{(1)}, \ldots, y^{(k)}]$ is called a (k)-*subsimplex* or a (k)-*face* of T if there are indices $0 \leq i_0 < i_1 < \cdots i_k \leq n$ such that $y^{(j)} = x^{(i_j)}$ for $0 \leq j \leq k$. ■

Obviously, the (0)- and (1)-faces of T are just its vertices and edges, respectively. The number of (k)-faces of an (n)-simplex T is $\binom{n+1}{k+1}$. Note that the previous relation is a trivial adjacency relation because all simplicial meshes verify it. The focus of this paper is the study of non-trivial relations and the study of their averages when a particular partition is recoursively applied to an initial mesh.

Definition 2.5 (skeleton) Let τ be any n-dimensional ($n = 2$ or 3) conforming triangular mesh. The *k-skeleton* of τ is the union of its k-faces. The $(n-1)$-skeleton is also call the skeleton [8]. ∎

For instance, the skeleton of a triangulation in three dimensions is comprised of the faces of the tetrahedra, and in two dimensions the skeleton is the set of the edges of the triangles. It should be noted however, that the skeleton can be understood as a new triangulation: if τ is a 3-D conforming triangulation in $\mathbb{R}^3$, $skt(\tau)$ ia a 2-D triangulation embedded in $\mathbb{R}^3$. Furthermore, if τ is conforming, then $skt(\tau)$ is also conforming.

Note that if we define some simplex partition over a conforming triangulation in which every element is divided into the same number of son-elements, and this hold also for the skeleton elements, the iterative application of such partition to a conforming triangulation always yield in another (finer) conforming triangulation.

Definition 2.6 (skeleton-regular partition) For any triangle or tetrahedron t, a partition of t will be called *skeleton-regular* if the following properties hold:

1. All the topological elements of the same dimension, that is all the elements of the k-skeleton ($0 \leq k < n$) are subdivided in the same number of son-elements.

2. The meshes obtained by application of the partition to any individual element in any conforming triangulation are conforming. ∎

Definition 2.7 (constitutive equations) When a skeleton-regular partition is applied to any initial mesh, there exist recurrence relations between the number of topological elements in the refined mesh and the number of topological elements in the unrefined mesh. These recurrence equations will be called *constitutive equations* of the partition. ∎

In the following we shall show that different partitions may have the same constitutive equations, and consequently they have the same asymptotic average numbers of topological adjacencies.

Definition 2.8 (equivalent partitions) Two partitions of the same element will be called *equivalent in average* or *topologically equivalent in average* if they have the same constitutive relations. ∎

3 Skeleton-regular partitions

In this section we introduce some of the most well known skeleton-regular partitions in two and three dimensions. All of them are based on edge bisection.

3.1 2-D Skeleton-regular partitions

Definition 3.1 (4T similar partition) The original triangle is divided into four son-triangles by connecting the midpoints of the father-triangle by straight line segments parallel to the sides. ■

Note that according with the definition, all the triangles are similar to the original one (see Figure 1(a)). This is one of the simplest partitions of triangles considered in literature (see for example [9, 10]). Bey notes in [11] that this is the 2-D version of the Freudenthal's algorithm [12].

Definition 3.2 (4T-LE partition) 4T partition bisects the triangle also in four triangles as it is shown in Figure 1(b). The triangle is first subdivided by its longest edge, and then the two resulting triangles are bisected by the midpoint of the common edge with the original triangle. In the following this partition will be called 4T-LE partition. ■

Longest-edge partitions have been studied by Rivara (see, for instance [13, 14]). 4T-LE partition is also very similar to the *newest-vertex insertion* method by Mitchell [15]. In fact, in many cases, depending on the geometry of the initial 2D triangulation, if the longest edge is chosen as the refining edge in the newest-vertex insertion method, then both are equivalent.

Definition 3.3 (2-D Baricentric partition) For any triangle t the baricentric partition of t is defined as follows:

1. Put a new node P at the baricentric point of t, and put new nodes at the midpoints of the edges.

2. Join the baricentric point P with the vertices of the edges, and with the nodes at the midpoints of the edges. (See Figure 1(c)). ■

Definition 3.4 (4T-SE partition) It is the partition in four triangles in which the shortest-edge of the initial triangle is chosen to perform the first bisection, and then we proceed as in the 4T-LE partition. (See Figure 1(d)). ■

It is worth pointing out here that in the similar and in the baricentric partition all the edges play the same role at parititioning the triangle, but this is not true for the 4T-LE partition, and the 4T-SE partition, in which the longest and the shortest edges are respectively distinguished edges.

Remark 3.1 *Note that different partitions can have the same recurrence associated equations, because these equations depend only on the number of son-elements for each particular original element. For example, 2-D similar partition, 4T-LE partition and 4T-SE partition have all the following constitutive equations:*
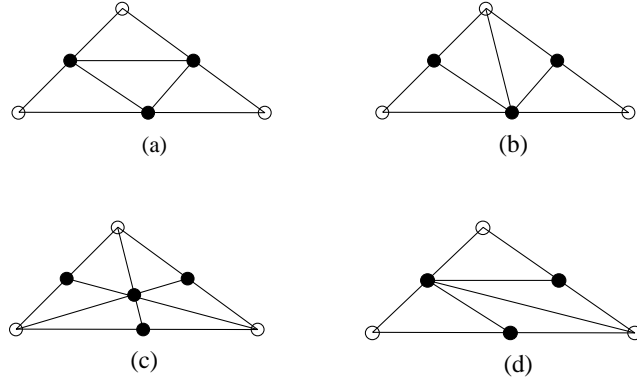
Figure 1: Four partitions in 2D

$$
\begin{aligned}
N_n &= N_{n-1} + E_{n-1} \\
E_n &= 2 \times E_{n-1} + 3 \times F_{n-1} \\
F_n &= 4 \times F_{n-1}.
\end{aligned}
$$
(3)

where N_n, E_n, and F_n are respectively the numbers of nodes, edges, and triangles in the mesh obtained after n applications of global refinement, that is after partitioning n times all the elements of the initial mesh.

In general an skeleton-regular simplex partition in 2-D will have the following constitutive equations:

$$
\begin{aligned}
N_n &= N_{n-1} + a \times E_{n-1} + b \times F_{n-1} \\
E_n &= c \times E_{n-1} + d \times F_{n-1} \\
F_n &= e \times F_{n-1}.
\end{aligned}
$$
(4)

where a is the number of nodes per edge, b the number of internal nodes per triangle, c the number of son-edges per edge, d the number of internal edges per triangle, and e the number of son-triangles per triangle. Note that these parameters determine the skeleton-regular partition. We shall prove that the asymptotic average relations are independent of these parameters, and hence all the 2-D skeleton-regular partitions show the same behavior in the limit of the average adjacencies, that is all the 2-D skeleton-reguar partitions are topologically equivalent in average.

3.2   3-D skeleton-regular partitions

In three dimensions several techniques have been developed in the last years for refining (and coarsening) tetrahedral meshes. A general overview can be found in [11]. Algorithms based on simple longest edge bisection have been developed by Rivara and Levin [16], and by Muthukrisnan *et al.* [17]. This refinement partition is not considered in this work since applying the longest-edge bisection to all

the tetrahedra to an initial mesh yields in general a not-confoming new mesh, and because if the LE partition is recursively applied until the mesh become conforming, not all the edges are subdivided in the same number of son-edges. Here, we resume briefly the main partitions in eight sub-tetrahedra.

Definition 3.5 (3-D Freudenthal-Bey partition) The original tetrahedron is divided into eight son-tetrahedra by cutting off the four corners by the midpoints of the edges (Figure 2) and the remaining octahedron is subdivided further in one of the three different ways corresponding to one of three possible interior diagonals (Figure 3) [11, 12]. This interior diagonal has to be choosen carefully in order to satisfy the stability condition, that is the non-degeneracy of the partition. ■
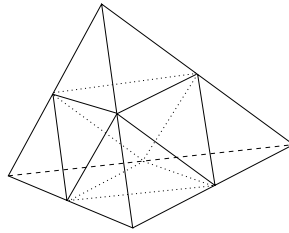
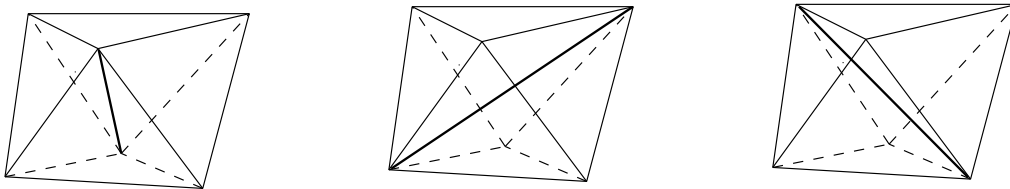Figure 2: First step of 3-D Freudenthal-Bey partition

Figure 3: The three possibilities for dividing the interior octahedron into four tetrahedra

Bey has proved recently [11] that this partition presents an optimal number of congruence classes generated by recursive refinements, even in general dimension n.

Definition 3.6 (8T-LE partition) The original tetrahedron is divided into eight son-tetrahedra by performing the 4T-LE partition of the faces, and then subdividing the interior of the tetrahedra in a consistent manner with the performed division in the 2-skeleton, see [18, 19, 20]. ■

It has been proved [20] that the 8T-partition can be achieved by performing bisections by the midpoints of the edges of the original tetrahedron after ordering the edges by their lenght.

Theorem 3.1 *For any tetrahedron t of unique longest-edge, the 8T-LE partition of t is getting by:*

    1. LE-bisection of t producing tetrahedra t_1, t_2.
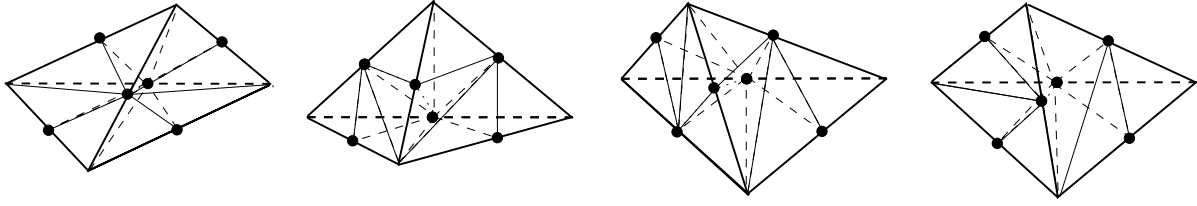
Figure 4: Refinement patterns for the 8T-LE partition

2. *Bisection of t_i, for $i = 1, 2$, by the midpoint of the unique edge of t_i which is also the longest edge of a common face of t_i with the original tetrahedron t, producing tetrahedra t_{ij}, for $j = 1, 2$.*

3. *Bisection of each t_{ij} by the midpoint of the unique edge equal to an edge of the original tetrahedron.*

It seams that this partition does not create a finite number of congruence classes when it is applied recoursively over an initial triangulation, and that the number of congruence classes depend on the geometry of the initial triangulation, in a similar way that it happens in 2D. However it has been sugested in [20], that the partition has a *self-improvement property* in the case of *bad shaped* triangulations.

Although 8T-LE partition is based on the length of the edges, and on the associated classification of the edges [18, 20] it yields in some cases to the same division of the tetrahedra as other patitioning patterns, like those by Bänsch [21], Kossaczký [22], or Liu and Joe [4]. It is not our aim here to compare with detail all these refinement algorithms. Note, however, that even in the cases in which the algorithms do not partitionate the mesh in the same way, these partitions are topologically equivalent, or topologically equivalent in average, since all divide equal dimension elements in the same number of son elements. They divide the edges into two edges, the triangular faces into four triangles, and the tetrahedra into eight son-tetrahedra.

Definition 3.7 (3-D Baricentric partition) For any tetrahedron t the baricentric partition of t is defined as follows:

1. Put a new node P at the baricentric point of t, put new nodes at the baricentric points of the faces of t, and put new nodes at the midpoint of the edges of t.

2. In each face of t do the baricentric triangular partition of the face. (See Figure 1(c) for the 2-dimensional case).

3. Join the baricentric point P with all the vertices of t, and with all the new nodes introduced before. (See Figure 5). ∎
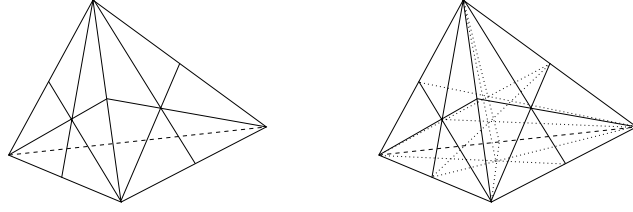
Figure 5: 3D Baricentric partition

4 Adjacency relations of the topological elements of the mesh

Beall and Shephard in [23] have studied the adjacency relations of the topological elements and have used it to analyze the storage requirements for the meshes commonly used in the finite element computations.

As we are going to restrict our study to simplex partitions in two and three dimensions, we do not need very sophisticated notations and definitions. In order to fix the manner in which the adjacency relations are going to be denoted, let us give an example:

The number of elements type E per each element of type F will be written as #(E *per* F).

We only distinguish between trivial and non-trivial adjacency relations, as follows:

Definition 4.1 A topological element t will be called *interior element* to some domain Ω if $interior(t) \subset interior(\Omega)$. In other case t is called exterior to Ω. ■

Definition 4.2 An adjacency relation will be called trivial if all interior elements verify the adjacency relation. ■

For example the number of edges per triangle is allways three, so the relation #(*edges per triangle*) = 3 is a trivial relation.

In 2D triangular partitions, the trivial adjacency relations are:

$$
\begin{aligned}
\#(\text{nodes per edge}) &= 2 \\
\#(\text{nodes per triangle}) &= 3 \\
\#(\text{edges per triangle}) &= 3 \\
\#(\text{triangles per edge}) &= 2
\end{aligned}
$$

In 3D tetrahedral partitions, the trivial adjacency relations are:

$$
\#(\text{nodes per edge}) = 2
$$

$$\begin{aligned}
\#(\text{nodes per face}) &= 3 \\
\#(\text{nodes per tetrahedon}) &= 4 \\
\#(\text{edges per face}) &= 3 \\
\#(\text{edges per tetrahedron}) &= 6 \\
\#(\text{faces per tetrahedron}) &= 4 \\
\#(\text{tetrahedra per face}) &= 2
\end{aligned}$$

Note that trivial relations are the number of k-faces of the j-simplices, where $k \leq j \leq n$, and $n = 2$ or $n = 3$. Besides there is another trivial relation: the number of n-simplices per $(n-1)$-face.

Definition 4.3 An adjacency relation will be called non-trivial if not all interior elements in all the possible meshes when a global partition in applied verify that relation. ∎

In the case of non-trivial adjacency relations we will consider the average of each adjacency relation over all the elements in the triangulation. For instance to denote the average of triangles per node we will write $Av\#(triangles\ per\ node)$.

5 Asymptotic results of the adjacency relations

Let τ_0 be an initial triangulation, in 2 or 3 dimensions, in which some skeleton-regular partition in recursively applied. If we note by N_0, E_0, F_0, and T_0, respectively the number of nodes, edges, faces and tetrahedra in τ_0 then the number of topological elements into the subsequent levels of meshes τ_n depend, by the constitutive equations, on the number of elements of the previous level of mesh τ_{n-1}. Besides, the average of the adjacency relations depend on the number of different topological elements in the mesh as the following lemmas establish.

Lemma 5.1

Let τ_n a 2-D triangulation with N_n nodes, E_n edges, and T_n triangles. Then, the non-trivial adjacency relations, in average, are:

$$\begin{aligned}
Av\#(triangles\ per\ node) &= \frac{3 \times T_n}{N_n} \\
Av\#(edges\ per\ node) &= \frac{2 \times E_n}{N_n}
\end{aligned}$$

and these two numbers are the same.

Proof Let us see first the average of triangles per node. Since we are calculating an average per node, the denominator has to be N_n. About the numerator, note that it should be

$$(5) \qquad \sum_{i=1}^{n} n_T(i),$$

where $n_T(i)$ represents the number of triangles sharing node i. That is, we have to add the number of triangles per each node, but this sum is equal to the sum of the number of nodes per triangle, so

$$(6) \qquad \sum_{i=1}^{n} n_T(i) = 3T_n.$$

Note that for the average number of edges per node the reasoning is the same. This follows from the fact that the average number of triangles per edge is $\#(triangles\ per\ edge) = 2 = \frac{3 \times T_n}{E_n}$, so $3 \times T_n = 2 \times E_n$, and the proof is completed. ∎

Lemma 5.2

Let τ_n a 3-D triangulation with N_n nodes, E_n edges, F_n faces, and T_n tetrahedra. Then, the non-trivial adjacency relations are:

$$
\begin{aligned}
Av\#(tetrahedra\ per\ edge) &= \frac{6 \times T_n}{E_n} \\
Av\#(faces\ per\ edge) &= \frac{3 \times F_n}{E_n} \\
Av\#(tetrahedra\ per\ node) &= \frac{4 \times T_n}{N_n} \\
Av\#(faces\ per\ node) &= \frac{3 \times F_n}{N_n} \\
Av\#(edges\ per\ node) &= \frac{2 \times E_n}{N_n}
\end{aligned}
$$

Proof The argument is the same as in Lemma 5.1. ∎

In order to calculate the asymptotic behavior of the average adjacencies of the topological elements of a particular skeleton-regular partition, we have to solve the constitutive equations associated to that partition. This can be done by means of generation functions [26] or even, by using a symbolic calculus package like MAPLE © [27].

The constitutive equations can be solved easily writing the equations in matricial form, if the associated matrix is diagonalizable [27], following this classic theorem [24]:

Theorem 5.1

Let $u_n = A^n u_0$ a difference equation, in which matrix A is diagonalizable, that is there exists a non-singular matrix S, such that $A = SDS^{-1}$ with D being a diagonal matrix. Then $u_n = SD^n S^{-1} u_0$.

Next we report the results relative to the partitions of Section 3.

Theorem 5.2

Let τ be a (conforming) triangular mesh. For any skeleton-regular partition let N_n, E_n, T_n be respectively the total number of nodes, edges, and triangles after the n-th partition application. Then the asymptotic average adjacency numbers of topological elements are independent of the particular partition of each triangle and these numbers are as follows.

$$\lim_{n \to \infty} Av\#(triangles \ per \ node) = \lim_{n \to \infty} \frac{3 \times T_n}{N_n} = 6$$

$$\lim_{n \to \infty} Av\#(edges \ per \ node) = \lim_{n \to \infty} \frac{2 \times E_n}{N_n} = 6$$

Proof The constitutive equations 4 associated to a general skeleton-regular partition in a 2D triangulation can be written in matrix form as follows:

$$(7) \quad u_n = \begin{pmatrix} N_n \\ E_n \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix} \cdot \begin{pmatrix} N_{n-1} \\ E_{n-1} \\ T_{n-1} \end{pmatrix} = A \cdot u_{n-1} = A^n \cdot u_0 = A^n \cdot \begin{pmatrix} N_0 \\ E_0 \\ T_0 \end{pmatrix}$$

where N_0, E_0, and T_0 are the initial values of the number of nodes, edges and triangles respectively. Note that matrix A is non-singular since $c = \#(\text{edges per edge}) > 1$ and $e = \#(\text{triangles per triangle}) > 1$. Furthermore, $c \neq e$, since $c = a + 1$, $b = 1 - e + d$. Finally, from Euler's relation for the vertices, edges and triangles applied to the first partition of a single triangle we get $2 \cdot d = 3 \cdot e - c$.

So, since $c \neq e$ and both are greater than 1, the matrix A defining the constitutive equation is diagonalizable, and hence we can apply Theorem 5.1, to get the following value for u_n:

$$(8) \quad u_n = \begin{pmatrix} N_n \\ E_n \\ T_n \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{1}{2}(1 + 3a + 2b)^n - \frac{3}{2}(a+1)^n\right) T_0 + \left(-1 + (a+1)^n\right) E_0 + N_0 \\ \left(\frac{3}{2}(1 + 3a + 2b)^n - \frac{3}{2}(a+1)^n\right) T_0 + (a+1)^n E_0 \\ (1 + 3a + 2b)^n T_0 \end{pmatrix}$$

Once the recurrence equations have been solved, taking limits we obtain:

$$\lim_{n\to\infty} \frac{3 \times T_n}{N_n} = \lim_{n\to\infty} \frac{3 \times (1 + 3a + 2b)^n T_0}{\left(1 + \frac{1}{2}(1 + 3a + 2b)^n - \frac{3}{2}(a+1)^n\right) T_0 + \left(-1 + (a+1)^n\right) E_0 + N_0} =$$

$$= \lim_{n\to\infty} \frac{3 \times (1 + 3a + 2b)^n T_0}{\left(\frac{1}{2}(1 + 3a + 2b)^n\right) T_0} = 6$$

$$\lim_{n\to\infty} \frac{2 \times E_n}{N_n} = \lim_{n\to\infty} \frac{2 \times \left(\frac{3}{2}(1 + 3a + 2b)^n - \frac{3}{2}(a+1)^n\right) T_0 + (a+1)^n E_0}{\left(1 + \frac{1}{2}(1 + 3a + 2b)^n - \frac{3}{2}(a+1)^n\right) T_0 + \left(-1 + (a+1)^n\right) E_0 + N_0} =$$

$$= \lim_{n\to\infty} \frac{2 \times \left(\frac{3}{2}(1 + 3a + 2b)^n\right) T_0}{\left(\frac{1}{2}(1 + 3a + 2b)^n\right) T_0} = 6$$

$\blacksquare$

In 3D the situation about the asymptotic behavior of the adjacency relations between the topological elements in the mesh is quite different. Now we obtain different values for the average limit depending on the particular partition considered. In the following we show the results for the partitions presented in Section 3.

For example in the 8T-LE partition the average limit for the adjacencies between the topological elements is presented in the following theorem:

Theorem 5.3

Let τ be a (conforming) initial tetrahedral mesh in which the 8T-LE partition is recursively applied. Then the asymptotic average non-trivial adjacencies are the following:

$$\lim_{n\to\infty} Av\#(\text{tetrahedra per edge}) = \lim_{n\to\infty} \frac{6 \times T_n}{E_n} = \frac{36}{7}$$

$$\lim_{n\to\infty} Av\#(\text{tetrahedra per node}) = \lim_{n\to\infty} \frac{4 \times T_n}{N_n} = 24$$

$$\lim_{n\to\infty} Av\#(\text{faces per edge}) = \lim_{n\to\infty} \frac{3 \times F_n}{E_n} = \frac{36}{7}$$

$$\lim_{n\to\infty} Av\#(\text{faces per node}) = \lim_{n\to\infty} \frac{3 \times F_n}{N_n} = 36$$

$$\lim_{n\to\infty} Av\#(\text{edges per node}) = \lim_{n\to\infty} \frac{3 \times E_n}{N_n} = 14$$

Proof Note first that the constitutive equations for the 8T-LE partition are:

(9)
$$\begin{aligned}
N_n &= N_{n-1} + E_{n-1} \\
E_n &= 2 \times E_{n-1} + 3 \times F_{n-1} + T_{n-1} \\
F_n &= 4 \times F_{n-1} + 8 \times T_{n-1} \\
T_n &= 8 \times T_{n-1}
\end{aligned}$$

or in matrix form:

$$
(10) \qquad u_n = \begin{pmatrix} N_n \\ E_n \\ F_n \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} N_{n-1} \\ E_{n-1} \\ F_{n-1} \\ T_{n-1} \end{pmatrix} = A \cdot u_{n-1}
$$

Matrix A is diagonalizable, and:

$$
(11) \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & \frac{7}{3} & -\frac{7}{6} & \frac{7}{6} \\ 0 & \frac{7}{3} & -\frac{7}{2} & \frac{7}{6} \\ 0 & 0 & -\frac{7}{3} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} -\frac{3}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & \frac{3}{7} & -\frac{9}{14} & \frac{11}{14} \\ 0 & 0 & -\frac{3}{7} & \frac{6}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Taking into account Theorem 5.1 the constitutive equations are solved and the solution is:

$$
(12) \quad \begin{pmatrix} N_n \\ E_n \\ F_n \\ T_n \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{6}8^n - 4^n + \frac{11}{6}2^n - 1\right) T_0 + \left(1 - \frac{3}{2}2^n + \frac{1}{2}4^n\right) F_0 + \left(-1 + 2^n\right) E_0 + N_0 \\ \left(\frac{7}{6}8^n - 3 \cdot 4^n + \frac{11}{6}2^n\right) T_0 + \left(-\frac{3}{2}2^n + \frac{3}{2}4^n\right) F_0 + 2^n E_0 \\ \left(2 \cdot 8^n - 2 \cdot 4^n\right) T_0 + 4^n F_0 \\ 8^n T_0 \end{pmatrix}
$$

where N_0, E_0, F_0, and T_0 are the number of nodes, edges, faces and tetrahedra in the initial triangulation. Taking limits in the appropriate quotiens as in Theorem 5.2 we obtain the asymptotic average adjacencies. ∎

Remark 5.1 *Since the 3D Freudenthal-Bey partition is equivalent in average to the 8T-LE partition, they have the same asymptotic adjacencies.*

Theorem 5.4
Let τ be a (conforming) initial tetrahedral mesh in which the baricentric partition is recursively applied. Then the asymptotic average adjacencies are the following:

$$
\lim_{n\to\infty} Av\#(tetrahedra\ per\ edge) = \lim_{n\to\infty} \frac{6 \times T_n}{E_n} = \frac{66}{13}
$$

$$
\lim_{n\to\infty} Av\#(tetrahedra\ per\ node) = \lim_{n\to\infty} \frac{4 \times T_n}{N_n} = 22
$$

$$
\lim_{n\to\infty} Av\#(faces\ per\ edge) = \lim_{n\to\infty} \frac{3 \times F_n}{E_n} = \frac{66}{13}
$$

$$\lim_{n\to\infty} Av\#(faces\ per\ node) = \lim_{n\to\infty} \frac{3 \times F_n}{N_n} = 33$$

$$\lim_{n\to\infty} Av\#(edges\ per\ node) = \lim_{n\to\infty} \frac{3 \times E_n}{N_n} = 13$$

Remark 5.2 *It is inmediate to proof that, in the same way that in 2D*

$$Av\#(triangles\ per\ node) = Av\#(edges\ per\ node),$$

in three dimensions we have that

$$Av\#(tetrahedra\ per\ edge) = Av\#(faces\ per\ edge)$$

6   Concluding remarks

In this communication we have shown the average adjacencies for skeleton-regular triangular and tetrahedral partitions. The study of the asymptotic behavior of the partitions based on recurrence equation system could be a clue in the proof of the non-degeneracy or stability properties of some local refinement algorithms in 3D and higher dimensions based on these partitions. This study can be applied to other polyhedral or polygonal partitions of the space, not only simplicial partitions.

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