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Geometric properties of the Four Triangles Longest-Edge partition

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SUMMARY

In this work we study some remarkable properties of the longest-edge based refinement algorithms for triangular unstructured meshes. The refinement algorithm considered in the present work is based on the skeleton structure of a triangular mesh and divides the elements following the Four Triangles Longest-Edge partition. The study also answers the important question of how the size of the triangulation is affected when local refinement is applied. We prove both theoretically and empirically that the propagation of a single triangle refinement asymptotically extends to a few neighbor adjacent triangles. We found the limits of the propagation using two metrics which are related to the Longest-Edge Propagation Path ($LEPP$). The geometric place where non terminal triangles are located within each refined mesh is also investigated and some properties are presented.

KEY WORDS: mesh refinement; $LEPP$; longest edge bisection

1. INTRODUCTION

Triangulating a polygon plays a central role in Computational Geometry, and is a basic tool in many other fields as for example in Computer Graphics, Finite Element Method, etc. [3], [7]. The problem can be formulated as follows: given N spatial points not in general position of

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a polygonal region, join them by non-intersecting straight line segments so that every region internal to the polygon is a triangle (usually known as triangular mesh). The problem can also be stated for three dimensions. A related problem that also finds a considerable interest is the refinement of a mesh. The refinement problem can be described as any technique for modifying a mesh involving the insertion of at least one additional vertex in order to produce meshes with desired features: non-degeneracy of the elements, conformity and smoothness. Degeneracy elements stand for the ocurrence of thin and long triangles, this can lead to undesirable behavior affecting numerical stability and producing visual artifacts. Conformity refers to the requirement that the intersection of non-disjoint triangles is either a common vertex or a common side. A vertex in the interior of an edge is called *non-conforming*. The smoothness condition states that the transition between small and large elements should be gradual. In general, there are two possible strategies for refining a mesh: (1) local refinement, when the process modifies a group of triangles and (2) global (also known as uniform) refinement, when all triangles in the mesh are chosen for refining. Many criteria have been studied as to what constitutes a 'good' triangulation [4], some of which involve maximizing the smallest angle, minimizing the total edge length or maximizing triangle perimeter and these are common criteria of both triangulation and refinement of meshes [1].

Several algorithms for the refinement of meshes have been developed in the last years. Algorithms based on the Delaunay criterion are very used because they construct triangulations assuring the criteria of the non-small angle [13] and producing nearly equiangular triangles. A very known Delaunay based algorithm is the Ruppert's algorithm [19] which comes with a strong theoretical guarantee: all new angles, that is, angles not present in the input mesh, are greater than 20^o, [3, 19]. Bern and Eppstein, [5], shown how to triangulate arbitrary polygons using a polynomial number of right and acute triangles. They provided two algorithms for the triangulation and refinement problems: given a polygon with n sides, the triangulation algorithm produce a new triangulation with $\mathcal{O}(n^2)$ non-obtuse triangles and for the refinement algorithm with $\mathcal{O}(n^4)$ non-obtuse triangles.

On the other hand, longest-side refinement algorithms [14, 15] guarantee the construction of good quality irregular triangulations. This is due to three facts: (1) the known bound on the small angles of the triangles generated: all angles in subsequent refined triangulations are greater than or equal to half the smallest angle in the input triangulation, (2) the smooth irregular triangulation obtained and (3) the locality of the refinement process. However, a questionable point of these algorithms as remarked by Rivara in [18] and Jones and Plassmann in [10] is: How does refinement propagation (to assure the conformity) affect the size of the triangulations? The expected answer should be: the minimum size such that the three previous conditions hold. The question can also be expressed as: How does the propagation extend to neighbor triangles in the mesh? The worst case is such that a single triangle refinement makes all of triangles to be also refined, [10], see Figure 1. On the other hand, a more acceptable answer is such that after refining a single triangle, the propagation extends only to a few neighbor triangles and besides the smoothness of the result is assured. Of course, the difference between these two situations depend on the geometry of the inital meshes. This paper copes with this question and provides both theoretical and empirical evidences showing that the iterative application of local or global refinement to an arbitrary unstructured triangular mesh produces meshes in which the propagation of a single element refinement is reduced in each stage, making it to tend to a fix constant.

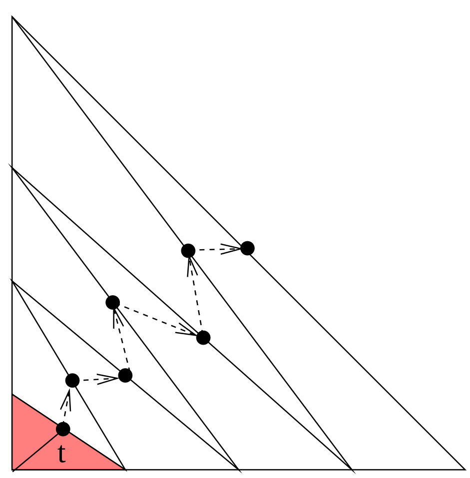


Figure 1. Longest edge refinement propagation. The dependencies in the propagation when refining t are indicated by arrows

2. PRELIMINARIES. THE REFINEMENT AND THE PROPAGATION PROBLEM

The refinement of triangular meshes involves two main tasks. The first task considers the partition of the target triangles and the second one the propagation or extension due to the conformity.

Several manners for partitioning triangles have been studied in last years. The simplest way to divide a triangle is the *Simple Bisection* which bisects the triangle into two subtriangles by connecting the midpoint of one of the edges to the opposite vertex. If the longest edge is chosen for the bisection, then it is called the *Longest Edge Bisection. Similar Partition* divides the triangle into four triangles by connecting the midpoints of the edges by straight line segments, producing four subtriangles which are similar to the original one. This partition has been widely used in Finite Element computations [8]. *Four Triangles Longest Edge Partition, (4T-LE)* bisects a triangle into four subtriangles where the triangle is first subdivided by its longest edge, and then the two resulting triangles are bisected by joining the new midpoint of the longest edge to the midpoint of the remaining two edges of the original triangle, see Figure 2 (d). Rivara first developed a refinement algorithm based on the *4T-LE* partition [15].

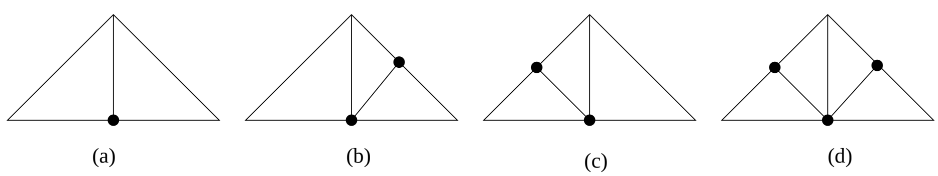


Figure 2. The four posible patterns in the 4T-LE refinement: (a) Bisection in two triangles, (b-c) Division in three triangles and (d) Division in four triangles

The second task in the refinement problem is to ensure the conformity of the mesh. For this purpose, it is necessary to determine additional irregular patterns which makes it possible to extend the refinement to neighbor triangles. Bank *et al.* in [2] developed a conformity strategy for the Similar Partition based on the simplest choice: connection of non-conforming nodes to the opposite vertices of adjacent triangles. This scheme has the drawback that the new divided triangles loose the good properties in terms of the interior angles and that the smoothness of the resulted mesh is not assured. Other strategy for the conformity process is based on the length of the edges. In this case, the possible patterns in the *4T-LE* refinement to guarantee the conformity can be seen in Figure 2. In this patterns, longest-edge of the non-conforming triangles is always divided and therefore the refinement area extends for conformity in a kind of 'domino' effect. We concentrate in this paper on the study of the propagation due to longest edge refinement and for this purpose the following concepts and definitions are given in order:

Definition 1 (Longest edge neighbor) *The longest edge neighbor of a triangle t is the neighboring triangle t^* which shares with t the longest edge of t.*

In particular, when considering the Simple Bisection of a single triangle, the propagation extends following the longest edge neighbors of the triangle. Note that the longest edge neighbor, if exists, is unique.

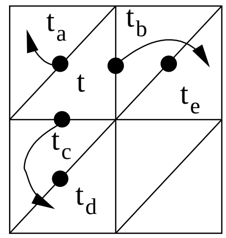
Definition 2 (Longest Edge Propagation Path (LEPP)) *The Longest Edge Propagation Path of a triangle t, as defined by Rivara in [15, 16], is the ordered list of all adjacent triangles $\Lambda = \{t_0 = t, t_1, \ldots, t_n\}$ such that t_i is the longest edge neighbor triangle of t_{i-1}.*

Note that the $LEPP$ provides the actual triangles to be refined when a given triangle is choosen for refinement. As a consequence, the $LEPP$ is the main structure of adjacent triangles used by the algorithms. In [20, 21] is presented a version of the refinement algorithm that uses an efficient data structure that explicity implements the $LEPP$. The $LEPP$ concept has also been used in combination with the Delaunay criterion to present a local refinement algorithm, in [17].

Definition 3 (Exterior and interior triangle) *Let τ be a two dimensional triangular mesh of a bounded domain Ω. A triangle $t \in \tau$ is said to be exterior if t has an edge belonging to the boundary of Ω, $\partial\Omega$. Otherwise, we say that t is an interior triangle of τ.*

Definition 4 (Terminal Triangles) *Let Λ be the LEPP of a given triangle in any triangular mesh τ. We say terminal triangles of τ to either:*

1. A single exterior triangle of longest edge in the boundary of the τ (single terminal triangle), or



Triangle	LEPP
t	$\{t,t_a\}$
t_a	$\{t_a,t\}$
t_b	$\{t_b,t_e\}$
t_c	$\{t_c,t_d\}$

Figure 3. LEPP's resulted in the 4T-LE refinement of t

2. A pair of adjacent triangles sharing the longest edge (pair of terminal triangles).

Terminal triangles are the last triangles in the ordered path of triangles in *LEPP's*, [16, 17]. As the ocurrence of single terminal triangles is only restricted to the boundary of a mesh, we shall consider in this paper the pair of terminal triangles. For this reason, in the following we shall refer to the *pair of terminal triangles* as *terminal triangles.*

If all the triangles in a mesh are terminal, this leads to a considerable simplification in $LEPP$'s structures, since all the $LEPP$ lists are comprised of two triangles (a pair of terminal triangles). For example, in Figure 3, for a given interior triangle, the $LEPP$ is comprised exactly of two triangles.

Next definition introduces the concept of *balanced mesh* in relation with the amount of terminal triangles in a mesh.

Definition 5 (Balanced mesh) *Let τ be a two dimensional triangular mesh, τ is said to be balanced if it is comprised of terminal triangles. In other case is said to be a non-balanced mesh.*

If τ is a non-balanced mesh, the mesh obtained by the application of uniform *4T-LE* partition to the mesh τ is comprised of terminal triangles and some other triangles.

To measure the ratio between terminal triangles and the total number of triangles in a mesh τ, which is relevant to know how balanced is a mesh, we define the *balancing degree* as follows:

Definition 6 (Balancing degree) *Let τ be a two dimensional triangular mesh containing T triangles and TT terminal triangles. Then, the balancing degree of τ, noted as $B(\tau)$, is defined as follows:*

$$B(\tau) = \frac{TT}{T} \tag{1}$$

Note that $0 \leq B(\tau) \leq 1$ and in the case that $B(\tau) = 1$, then the mesh is balanced. It should be noted that TT means the number of triangles belonging to some pair of terminal triangles, so, single terminal triangles are not considered in this formula.

Proposition 1. *If a mesh τ is such that the balancing degree is 0, then the conformity process when refining any triangle $t_0 \in \tau$ extends to the boundary of τ.*

Proof:

If $B(\tau) = 0$, then for any triangle $t_0 \in \tau$, $LEPP(t_0) = \{t_0, t_1, ..., t_n\}$ where t_n is an exterior triangle of τ with its longest-edge in $\partial\Omega$. Therefore, when t_0 is refined, the refinement extends until t_n is also divided. See Figure 1. $\square$

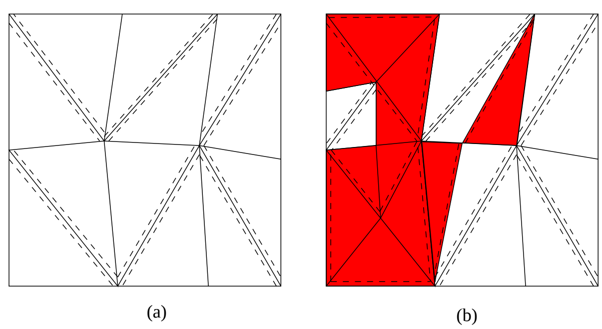


Figure 4. (a) Balanced mesh, (b) Non-balanced mesh (shaded triangles are non terminal)

Figure 4 shows a balanced mesh in (a) and a non-balanced mesh in (b). From now on we represent with a dashed line the longest-edge of triangles in a mesh.

A typical example of balanced mesh is the mesh comprised entirely of rectangular triangles sharing the longest-edges. In such a mesh, if one applies uniform *4T-LE* refinement, then only similar triangles to the original ones are obtained, and all of them are terminal.

3. PROPAGATION PROPERTIES OF THE ITERATIVE 4T LONGEST EDGE REFINEMENT

Next we introduce the *Conformity Neighborhood* associated with the application of the *4T-LE* local refinement. This concept will be of help in the study of the propagation properties.

Definition 7 (Conformity Neighborhood V_c) *Let τ be a triangular mesh. When refining a triangle $t \in \tau$, the Conformity Neighborhood of t, V_c(t), is the set of triangles that also need to be refined due to the conformity process for t.*

Definition 8 (M1) *Let τ be a triangular mesh. When refining a triangle $t \in \tau$, $M1(t)$ is said to be the cardinal of $V_c(t)$: $M1(t) = |V_c(t)|$.*

Following proposition state the relationship between $M1$ and *LEPP*:

Proposition 2. *Let τ be a triangular mesh. For each $t \in \tau$,* M1(t) *is the sum of the lengths of the $LEPP's$ of the neighbors of t, (excluding triangle t).* □

Figure 1 shows that always is possible to construct a mesh in which $M1(t)$ is as big as we want. In fact, in a mesh as that in Figure 1, the average of $M1$ is of linear order in the number of elements, N: $\mu(M1) = \frac{\sum M1(t)}{T} = \frac{\sum_0^{N-1} k}{N} = \frac{\frac{N-1}{2} \cdot N}{N} = \frac{N-1}{2}$. On the other hand, if $B(\tau) = 1$ as in Figure 3, $M1(t) \leq 5\ \forall t \in \tau$.

Definition 9 (M2) *Let τ be a triangular mesh. For each $t \in \tau$,* M2(t) *is the maximum length of the $LEPP's$ of the neighbor triangles of $t \in \tau$, (excluding triangle t).* □

It should be noted that when refining a triangle $t \in \tau$, the conformity process extends at most by the three edges of t and the propagation defines at most three lists of ordered triangles. $M2(t)$ is the maximum number of triangles of the three resulted lists. For example, in Figure 3, $M2(t) = 2$ because the maximum number of triangles among $\{t_b, t_c\}$, $\{t_c, t_d\}$, $\{t_a\}$ is 2. $M2$

can be viewed as the radius of $V_c(t)$.

Both parameters are relevant in the computational and storage cost of longest edge refinement algorithms. *M1* and *M2* are also related to balanced meshes, as the following proposition establishes:

Proposition 3. *Let τ be a balanced mesh with at least 6 triangles. Then, for each interior triangle $t \in \tau$, $M1(t) = 5$ and $M2(t) = 2$. Moreover, $M1(t) = 5 \iff M2(t) = 2$, and if $M1(t) = 5$ for all interior triangle $t \in \tau$, then the submesh of interior triangles of τ is balanced.*

Proof:
In a balanced mesh all the triangles are terminal. In such a mesh, a given interior triangle t is adjacent to other triangle by their common longest edge, see Figure 4 (a) and each of these pairs of terminal triangles represents a particular $LEPP$. Then, if one consider the propagation of the refinement by the $LEPP$ of a single interior triangle, the sum of the triangles refined in the propagation is exactly 5 ($M1(t) = (2+2+1) = 5$), see for example Figure 3. On the other hand, $M2(t)$ is equal to 2 because, the $LEPP$ exactly contains a pair of terminal triangles. □

Our next goal is to prove that the uniform application of the *4T-LE* partition (also discussed for local refimement later) will produce a series of meshes with increasing balancing degree approaching 1. As a consequence, we also shall prove that the mean of $M1$ and the mean of $M2$ tend to 5 and 2 respectively, when the number of refinements applied tends to infinity. These properties are related with the number of distinct similar triangles that appear when the *4T-LE* partition is uniformly applied.

Proposition 4. *[14] (a) The first application of the 4T-LE partition to a given triangle t_0 introduces two new triangles which are similar to the original triangle t_0 (moreover, these two triangles are exterior by their longest edges) and two (potentially) new similar triangles.*
(b) The iterative application of the 4T-LE partition to a given triangle t_0 introduces at most one new distintic (up to similarity) triangle in each iteration. □

Proposition 5. *If the 4T-LE partition to an initial triangle t_0 introduces two new similar* *triangles t_1 sharing their longest edge, then the iterative application of the 4T-LE partition introduces pairs of terminal triangles excepting the triangles located at the longest edge of t_0. Moreover, in this case only two classes of similar triangles are generated, t_0 and t_1, see Figure 5.*

Proof:
The situation of the hipothesis is depicted in Figure 5 (a). Considering parallelism we get the result, see Figure 5 (b). □

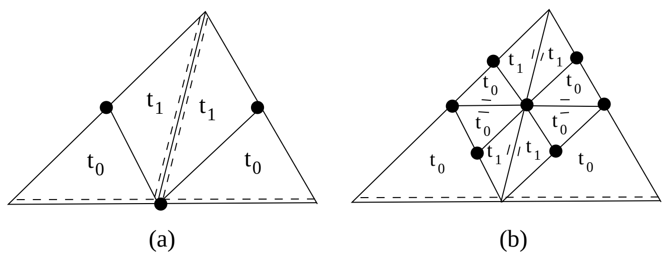


Figure 5. 4T-LE partition and generation of terminal triangles

To demonstrate that the *4T-LE* refinement introduces meshes with relatively more terminal triangles for any arbitrary triangular mesh we consider the distintic types of triangles: rectangular, acute and obtuse, and separated treatments are considered. We begin in the next Proposition with the rectangular and acute triangle cases:

Proposition 6. *(Rectangular and acute triangle cases) The application of the 4T-LE partition to an initial rectangular or acute triangle t_0 produces two new triangles similar to the original one (located at the longest edge of t_0) and a pair of terminal triangles. These triangles are also similar to the original one t_0 in the case of rectangular tirangle t_0, and they are similar to each other but possibly non-similar to the initial one in the case of acute triangle t_0. See Figures 6 and 7.* □

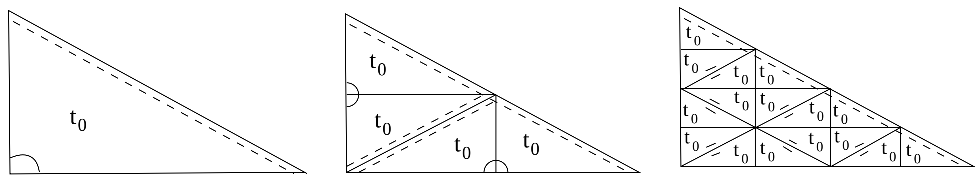


Figure 6. 4T-LE partition. Rectangular triangle

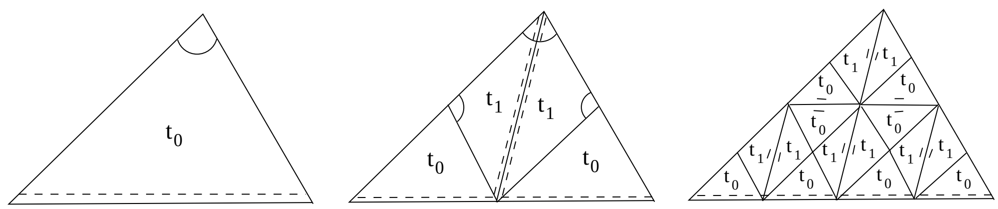


Figure 7. 4T-LE partition. Acute triangle

The obtuse triangle case offers a different situation:

Proposition 7. *(Obtuse triangle case) The application of the 4T-LE partition to an initial obtuse triangle t_0, produces two new triangles similar to the original one (located at the longest edge of t_0) and for the other two new generated triangles, t_1, it is possible one of the following cases:*

1. *To be a pair of similar terminal triangles.*
2. *To be a pair of similar non terminal triangles.*

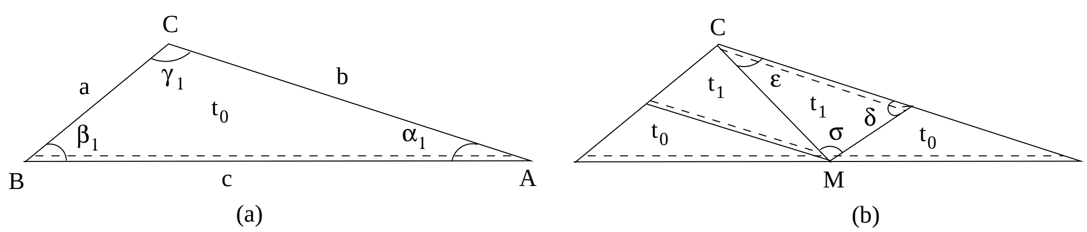


Figure 8. (a) Obtuse triangle t_0 (b) 4T-LE partition of t_0

Proof:

Let $\alpha_1 \leq \beta_1 \leq \gamma_1$ be the angles of the initial obtuse triangle t_0 and let a, b, c be the sides of t_0 respectively opposite to α_1, β_1 and γ_1. For the new non-similar triangles generated, we note ϵ the opposite angle to $\frac{a}{2}$ and σ the opposite angle to $\frac{b}{2}$. Since $\frac{a}{2} \leq \frac{b}{2}$ and $\epsilon \leq \sigma$ the longest edge of t_1 is either the new edge CM or $\frac{b}{2}$. In the first situation (point 1 of the Proposition), triangles t_1 are a pair of terminal triangles sharing edge CM as the longest edge. Note that in this case the longest angle of triangles t_1 are opposite to edge CM and since $\delta + \gamma_1 = \pi$, then $\delta < \frac{\pi}{2}$, so triangles t_1 are acute. Besides, the subsequent *4T-LE* partition of triangles will not produce new non-similar triangles (see Proposition 5 and Figure 5).

In the second case, the longest angle of t_1 is σ (see triangles t_1 in Figure 8). The new triangles t_1 are not pair of terminal triangles (point 2 of the Proposition). Moreover, the application of *4T-LE* partition to triangles t_1 may produce a new pair of non-similar triangles t_2. $\square$

In the following we shall call to those triangles pointed out in point 1 of the Proposition 8, *Type 1 obtuse triangles* and for those pointed out in point 2 of the Proposition 8, *Type 2 obtuse triangles.*

It should be noted that the *4T-LE* partition always produces two new triangles similar to the original one (located at the longest edge of t_0) and excepting for Type 2 obtuse triangles, a pair of terminal triangles (similar or non similar to the original one). Moreover, in this scenario, the non terminal triangles generated by the iterative *4T-LE* partition are those located at the longest edge of the initial triangle, Proposition 5, see Figures 6 and 7.

The following Proposition states the auto-improvement property of the iterative *4T-LE* partition for obtuse triangles [14]:

Proposition 8. *If the 4T-LE partition of an obtuse triangle t_0 introduces a pair of similar non terminal triangles, (Type 2 obtuse triangles), then the new angles of the new triangle t_1 hold:*

1. $\gamma_2 = \sigma$

2. $\gamma_2 = \gamma_1 - \epsilon \leq \gamma_2 - \alpha_1$

where γ_i is the longestt angle in the triangle t_i and α_1 the smallest angle in the triangle t_0.
□

It is worth noting that in the *4T-LE* iterative refinement, Type 2 obtuse triangles are less obtuse than in the preceding mesh, and after a finite number of *4T-LE* partitions this leads to the case in which the new generated triangles will be no longer obtuse. Hence, when the generated triangles are non obtuse, it is then applied Proposition 6, (rectangular or acute triangle cases).

In agreement with the previous analysis, we state the next main results:

Proposition 9. *Let τ_0 be an initial triangular mesh and let $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ the sequence of nested meshes obtained by uniform application of 4T-LE partition to the previous mesh. Then, the balancing degree of the meshes tends to 1 when the number of iterative refinement grows to infinity, that is:*

$$\lim_{n\to\infty} B(\tau_n) = 1 \tag{2}$$

Proof:
It is enough to prove the result for the case in which the initial mesh τ_0 only contains a unique triangle t_0. Then, the number of generated triangles associated to the *4T-LE* partition in the n stage of refinement is:

$$T_n = 4^n \tag{3}$$

Firstly, we shall prove the proposition for initial rectangular, acute, and the Type 1 obtuse triangles. In this situation, the number of terminal triangles generated in the n stage of uniform *4T-LE* partition, TT_n, holds the following recurrence relation, (see Proposition 5 and Figure 5):

$$TT_n = 4TT_{n-1} + 2(T_{n-1} - TT_{n-1}) \tag{4}$$

with respective initial conditions $T_0 = 1$ and $TT_0 = 0$.

Equation (4) can be solved by writing Equations (3) and (4) in matricial form, since the associated matrix is diagonalizable [11]:

$$\begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}^n = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 4^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^n & -2^n + 4^n \\ 0 & 4^n \end{pmatrix}$$

which hold:

$$\begin{pmatrix} TT_n \\ T_n \end{pmatrix} = A^n \begin{pmatrix} TT_0 \\ T_0 \end{pmatrix} \tag{5}$$

From (5) it is obtained the following solution to Equation (4):

$$TT_n = 2^n TT_0 + (4^n - 2^n)T_0 = 4^n - 2^n \tag{6}$$

Therefore,

$$\lim_{n\to\infty} B(\tau_n) = \lim_{n\to\infty} \frac{TT_n}{T_n} = \lim_{n\to\infty} \frac{4^n - 2^n}{4^n} = 1$$

To complete the proof, we now consider the case of an initial Type 2 obtuse triangle t_0. Table I presents the number of distintic types of triangles generated by the *4T-LE* iterative refinement of t_0. We denote with t_j^n the number of triangles of similarity class t_j in the stage n of refinement. For example, after second refinement are generated 4 triangles similar to t_0, 8 triangles similar to t_1 and 4 new triangles similar to t_2.

Table I. Triangles evolution in the 4T-LE partition

Ref.	0	1	2	3	4	$\cdots$ n
t_0	1	2	4	8	16	t_0^n
t_1		2	8	24	64	t_1^n
t_2			4	24	96	t_2^n
t_3				8	64	t_3^n
t_4					16	t_4^n
$\cdots$						
t_n						t_n^n

From Proposition 7 (2) and Figure 8 we derive Table I, in which the following relation holds:

$$t_j^n = 2t_{j-1}^n + 2t_{j-1}^{n-1} \tag{7}$$

Equation (7) with initial condition $t_0^0 = 1$ can be easly expressed in terms of binomial coefficients as follows:

$$t_j^n = 2^n \binom{n}{k} \tag{8}$$

On the other hand, from Proposition 8, the iterative *4T-LE* partition of any obtuse triangle t_0 produces a finite sequence of distintic (up to similarity) triangles, t_j^i, $0 < j \leq k$. After that, there will be no longer distintic new generated triangles from other already generated, (see proof of Proposition 7). So, the number of terminal triangles TT_n after the k refinement stage with $n > k$ hold:

$$TT_n \geq 2^n \sum_{m=k}^{n} \binom{n}{m}$$

Hence, we can state that the balancing degree of the generated meshes, $B(\tau_n)$ verifies:

$$1 \geq B(\tau_n) \geq \frac{2^n \sum_{m=k}^{n} \binom{n}{m}}{2^n \sum_{m=0}^{n} \binom{n}{m}} = \frac{\sum_{m=k}^{n} \binom{n}{m}}{2^n}$$

Taking limits:

$$1 \geq \lim_{n\to\infty} B(\tau_n) \geq \lim_{n\to\infty} \frac{2^n \sum_{m=k}^{n} \binom{n}{m}}{2^n \sum_{m=0}^{n} \binom{n}{m}} = \lim_{n\to\infty} \frac{\sum_{m=k}^{n} \binom{n}{m}}{2^n}$$

To calculate this limit we take into account that:

$$\sum_{m=k}^{n} \binom{n}{m} = 2^n - \sum_{m=0}^{k-1} \binom{n}{m} \geq 2^n - \binom{n}{k-1}(k-1)$$

Therefore:

$$1 \geq \lim_{n\to\infty} B(\tau_n) \geq \lim_{n\to\infty} \frac{2^n - \binom{n}{k-1}(k-1)}{2^n} = 1$$

So, $\lim_{n\to\infty} B(\tau_n) = 1$. □

Corollary 1. *The iterative application of the 4T-LE uniform refinement to an initial triangular mesh τ_0 makes the mean of $M1$ and the mean of $M2$ tend to 5 and 2 respectively, when the number of refinements applied grows to infinity.*

Proof:

This result is directly derived from Propositions 3, and 9. □

The previous study is mainly concerned with the uniform refinement of meshes. The following propositions addresses the relationship of the previous study but to the local refinement problem.

Proposition 10. *Let τ_0 be an initial triangular mesh of bounded domain Ω and $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a sequence of locally refined meshes. Let $\Omega_i \subset \Omega$ be a sub-region of refinement in τ_i, $1 \le i < n$. Then:*

1. $\mu(M1(\tau_n)) = 5$ *and* $\mu(M2(\tau_n)) = 2$
2. $\lim_{n\to\infty} B(\tau_n) = 1$.

Proof:

It can be noted that the local refinement is confined to sub-regions Ω_i of Ω. However, in each of these sub-regions it is applied uniform refinement and points 1 and 2 of the Proposition hold when iterative uniform refinement is performed in such sub-regions. Only the Conformity Neighborhood of triangles in Ω_i is refined by conformity with other 4T-LE patterns, see Figure 2 (a-c). Then, the iterative application of the local refinement to the initial mesh τ produces meshes with increasing balancing degree (increasing number of terminal triangles), see Proposition 9 and Corollary 1, and then:

1. $\mu(M1(\tau) = 5$ and $\mu(M2(\tau)) = 2$ and
2. $\lim_{n\to\infty} B(\tau_n) = 1$. □

The iterative application of uniform *4T-LE* refinement to an arbitrary triangular mesh reveals a geometric structure of the edges that appears of the intermediate meshes. It can be defined as follows:

Definition 10. *Let $\mathcal{S}(\tau)$ be the set of non terminal triangles in a triangular mesh τ. A polyline is a set of connected longest-edges of non terminal triangles in $\mathcal{S}(\tau)$, and the set of of polylines of τ is called the Cantor set of τ and represented by $\mathcal{C}(\tau)$.*

The interest of $\mathcal{C}(\tau)$ comes from the fact that the location of polylines in $\mathcal{S}(\tau)$ represents the geometric place where the non terminal triangles are located. Therefore, $\mathcal{C}(\tau)$ could be usefull for example for applying a post-processing to the generated meshes in which non terminal triangles could be turned in terminal triangles, or it could be usefull for applying local refinement by other partitions, as similar partition, simple bisection etc.

Following proposition summarizes some remarkable properties of $\mathcal{C}(\tau)$:

Proposition 11. *Let $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a sequence of nested meshes obtained by uniform* 4T-LE *refinement. Let τ_k the fisrt mesh of Γ in which there are no new classes of triangles (up to similarty). Then:*

1. The number of edges in the Cantor set $\mathcal{C}(\tau_i)$ is exactly the double number of edges in $\mathcal{C}(\tau_{i-1})$, $k+1 \leq i \leq n$.

2. The number of edges in $\mathcal{C}(\tau_i)$ is exactly the number of non terminal triangles in τ_i. Hence, the size of $\mathcal{C}(\tau_i)$ is of order $\mathcal{O}(\sqrt{N})$ being N the number of triangles in τ_i.

3. Excepting the case of Type 2 obtuse triangles, the Cantor sets $\mathcal{C}(\tau_i)$, $k \leq i \leq n$, are shape and length invariant over the respective meshes. We say that $\mathcal{C}(\tau)$ is stable.

4. Excepting the case of Type 2 obtuse triangles, the sets of edges in $\mathcal{C}(\tau_i)$ $k \leq i \leq n$ are located at the longest edges of non terminal triangles of mesh τ_k. □

4. NUMERICAL EXPERIMENTS

In this section we present numerical evidence showing that the practical behavior of the *4T-LE* partition is in concordance with the reported theory in this work, mainly Propositions 9, 11 and Corollary 1.

4.1. 4T-LE REFINEMENT. TRIANGLE CASES STUDY

Firstly, we treat three triangle cases, rectangular, acute and Type 2 obtuse triangles. To these initial meshes we apply seven stages of uniform 4T-LE refinement. The goal in this first experiment is to calculate the number of terminal triangles and compare to the amount of other triangles in each stage of the refinement. In Table II it is reported the number of terminal triangles compared to other triangles. For the three cases can be noted as at refinement stage seven, the number of terminal triangles is clearly larger than the other triangles and this is in agreement with Proposition 9. Besides, it is presented from Figure 9 to Figure 14 the respective meshes as they are refined and the respective Cantor sets, $\mathcal{C}(\tau)$ (it is indicated with shaded color the non terminal triangles in each refined mesh).

Table II. Terminal triangles for the three triangles cases

Refinement Stage	Terminal Triangles	Other triangles
0 (Rectangular triangle)	0	1
1	2	2
2	12	4
3	56	8
4	240	16
5	992	32
6	4032	64
7	16256	128
0 (Acute triangle)	0	1
1	2	2
2	12	4
3	56	8
4	240	16
5	992	32
6	4032	64
7	16256	128
0 (Type 2 obtuse triangle)	0	1
1	0	4
2	2	14
3	26	38
4	162	94
5	802	222
6	3586	510
7	15234	1150

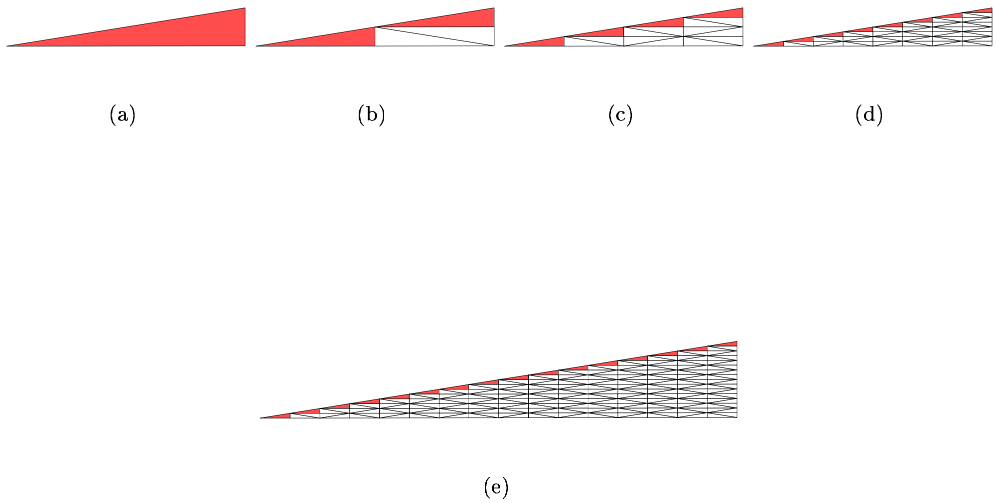


Figure 9. 4T-LE refinement. Rectangular triangle case

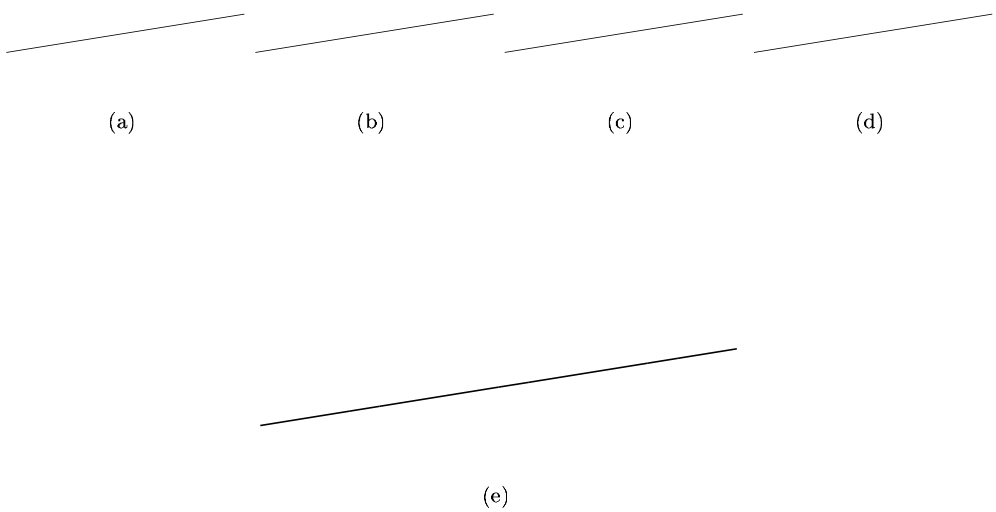


Figure 10. Cantor set in rectangular triangle case

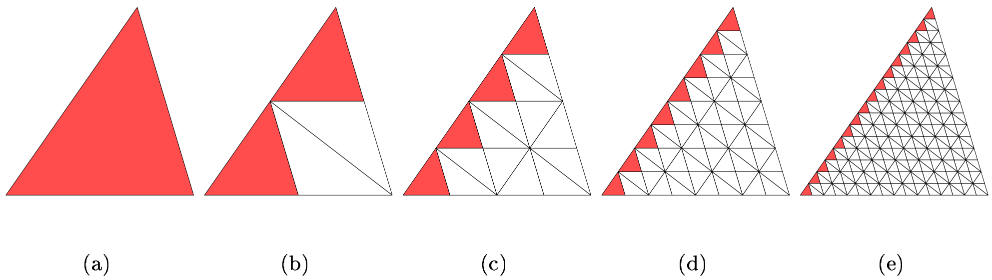


Figure 11. 4T-LE refinement. Acute triangle case

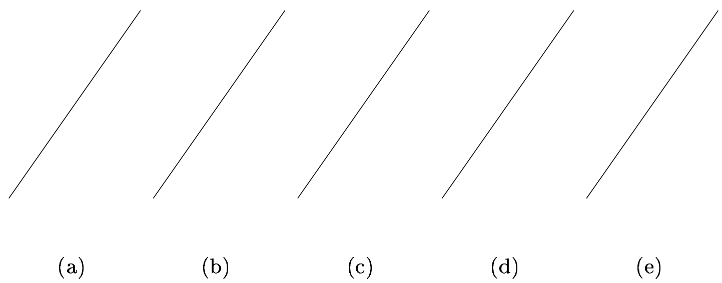


Figure 12. Cantor set in acute triangle case

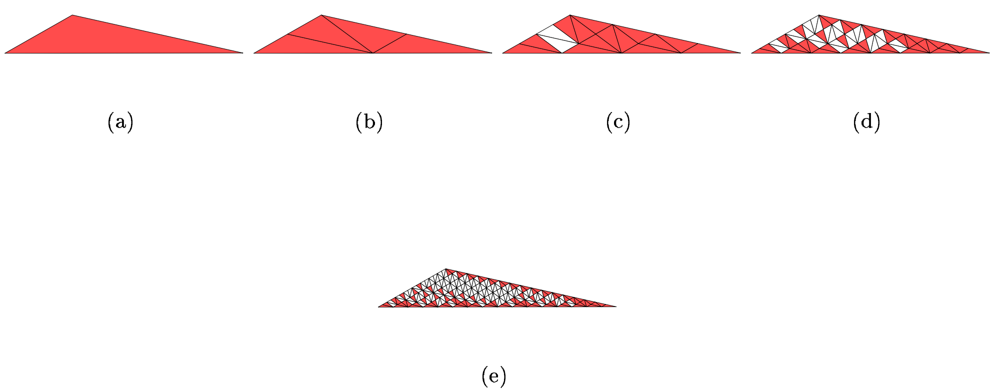
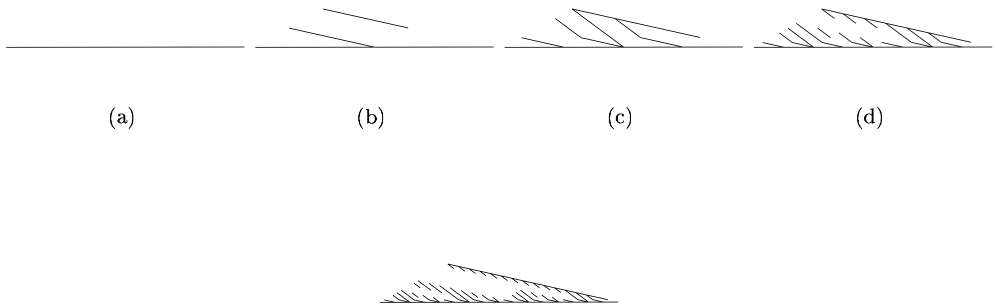


Figure 13. 4T-LE refinement. Obtuse triangle case



(a) (b) (c) (d)

(e)

Figure 14. Cantor set in obtuse triangle case

4.2. 4T-LE REFINEMENT. ARBITRARY MESHES STUDY

We next consider the examples of two initial meshes and four stages of uniform refinement. The mesh in Figure 15 (a) is a *Delaunay* initial mesh of square domain. The mesh in Figure 15 (b) is a bad-shaped arbitrary mesh, named for convenience, *pentagonal mesh.* It should be noted that the triangles in the Delaunay mesh are almost regular in terms of the angles and $B(\tau_0) = 0.4833$. On the other hand, the pentagonal mesh contains all triangles with increasing largest angles and decreasing smallest angles and $B(\tau_0) = 0$.

The refined meshes of the initial Delaunay mesh is presented in Figures 18 and 19. From Figure 19 (a)-(b) it can be distinguished the Cantor set $\mathcal{C}(\tau)$ of Definition 10. Tables III and IV report the $M1$ and $M2$ means. It is observed that both means tend to 5 and 2 respectively, as the refinement step increases.

In Tables V and VI the balancing degree, terminal triangles and total triangles in each refinement step are presented. It is observed from the last tables the increasing number of terminal triangles in the intermediate meshes (covering the area of the meshes), and as a result, the balancing degree tending to 1. In Figure 4.2 it is given a comparison between the balancing degrees of the two meshes. Note that in both meshes the balancing degree tends to 1 when the number of refinements increases, even in the pentagonal mesh, which exibits an initial balancing degree $B(\tau_0) = 0$.

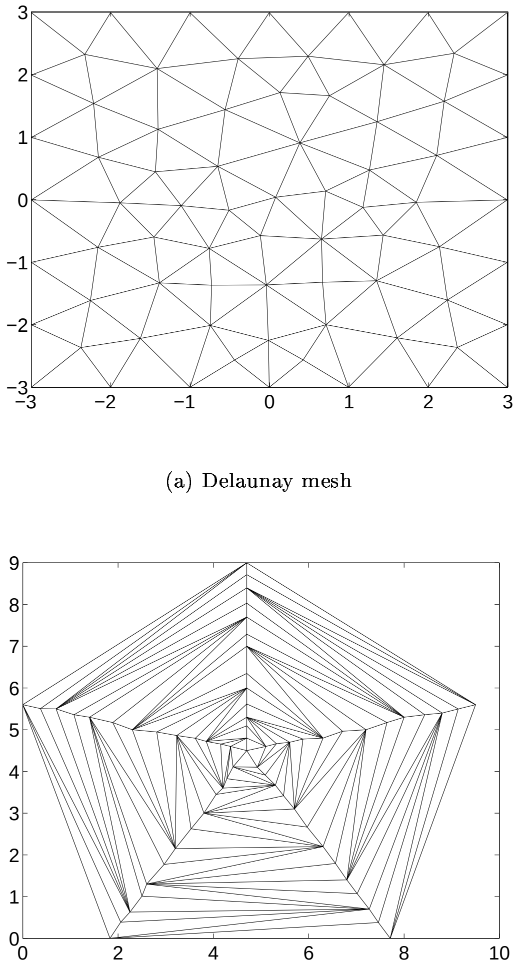
These results are also applicable to the local refinement scenario as described in Corollary 1. In order to demontrate evidence of this fact we consider the application of the *4T-LE* local refinement on a domain corresponding to the Gran Canaria Island. The initial mesh of this example is a Delaunay mesh and it is considered local refinement on disjoint subregions S_1, S_2 and S_3 with $S = S_1 \cup S_2 \cup S_3$, for innermost region S_3, intermediate region S_2 and outermost region S_1. Table VII and Figure 23 reveal the same behavior as in uniform refinement, this is, $M1$ and $M2$ tends to 5 and 2 respectively and the balancing degree aproaches to 1.

Table III. Delaunay mesh. M1 and M2

Ref. Step	Triangles	M1 Mean	M2 Mean
0 (Initial mesh)	120	5.2333	2.6250
1	480	5.1125	2.3813
2	1920	5.0573	2.1953
3	7680	5.0289	2.0967
4	30720	5.0145	2.0481
5	122896	5.0076	2.0286

Table IV. Pentagonal mesh. M1 and M2

Ref. Step	Triangles	M1 Mean	M2 Mean
0 (Initial mesh)	125	26.5440	14.3929
1	500	6.9100	3.8000
2	2000	6.2000	3.0485
3	8000	5.9976	2.8312
4	32000	5.4823	2.4123
5	128000	5.3706	2.2041



(a) Delaunay mesh

(b) Pentagonal mesh

Figure 15. Meshes for the test problem

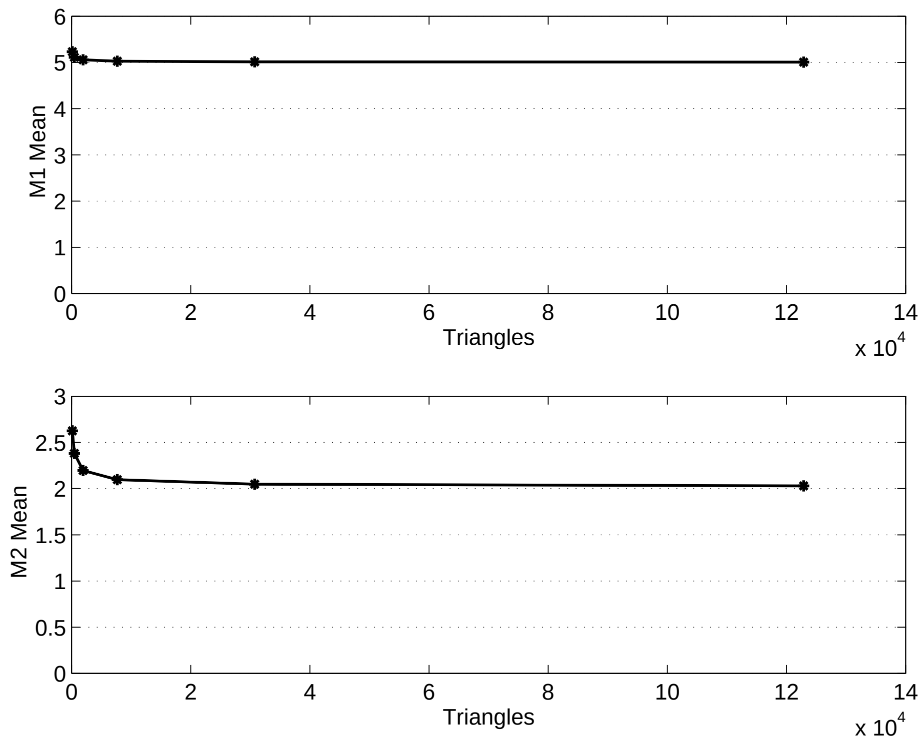


Figure 16. Delaunay mesh. Evolution of M1 and M2.

Table V. Delaunay mesh. Balancing degree

Ref. Step	Terminal triangles	Triangles	Balancing degree
0 (Initial mesh)	58	120	0.48333
1	356	480	0.74166
2	1672	1920	0.87083
3	7184	7680	0.93541
4	29728	30720	0.96770
5	121447	122896	0.98821

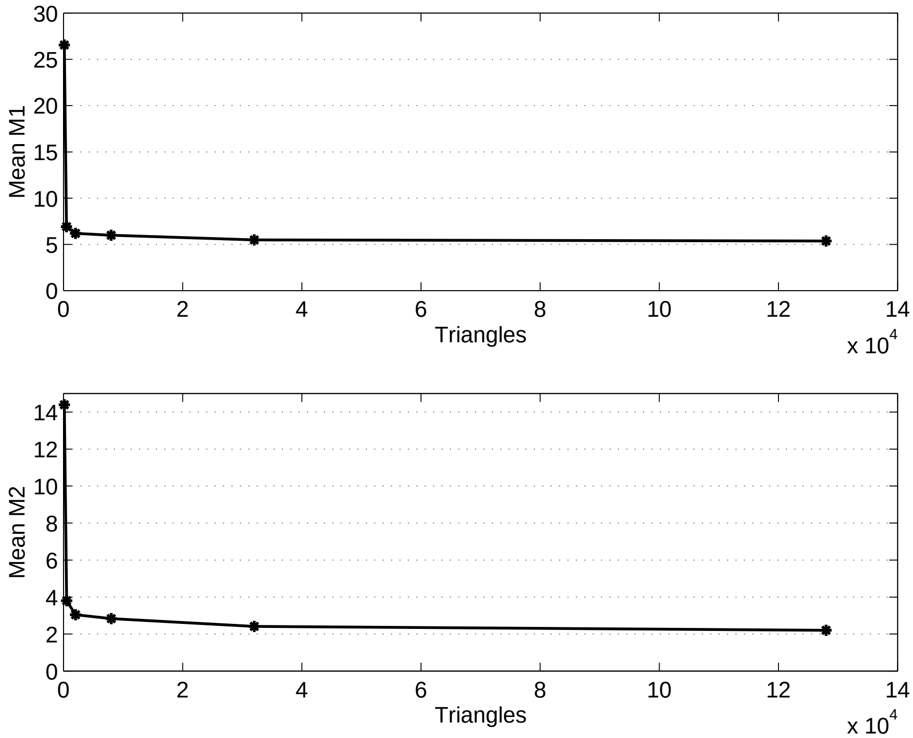
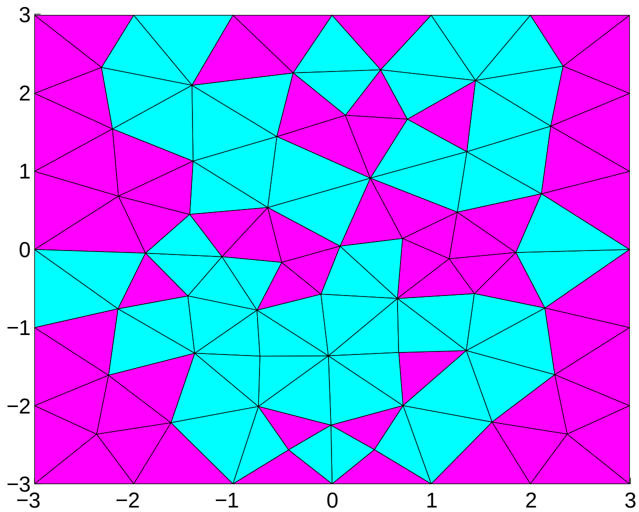


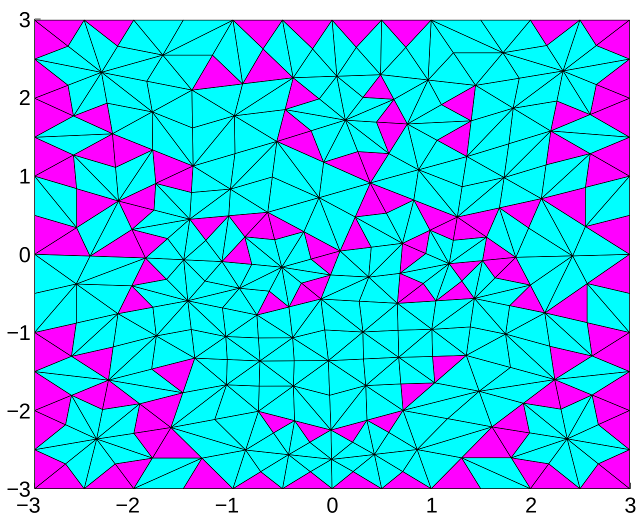
Figure 17. Pentagonal mesh. Evolution of M1 and M2

Table VI. Pentagonal mesh. Balancing degree

Ref. Step	Terminal triangles	Triangles	Balancing degree
0 (Initial mesh)	0	125	0
1	246	500	0.49200
2	1088	2000	0.54400
3	4778	8000	0.59725
4	21240	32000	0.66375
5	103970	128000	0.81230

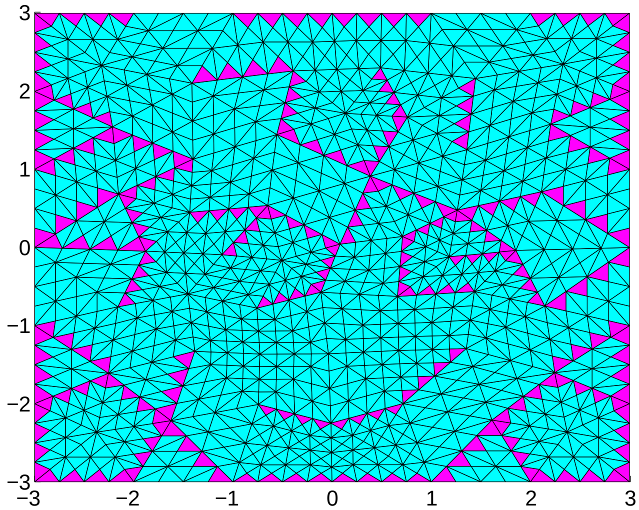


(a) Initial mesh, 120 triangles, 58 terminal triangles (blue color)

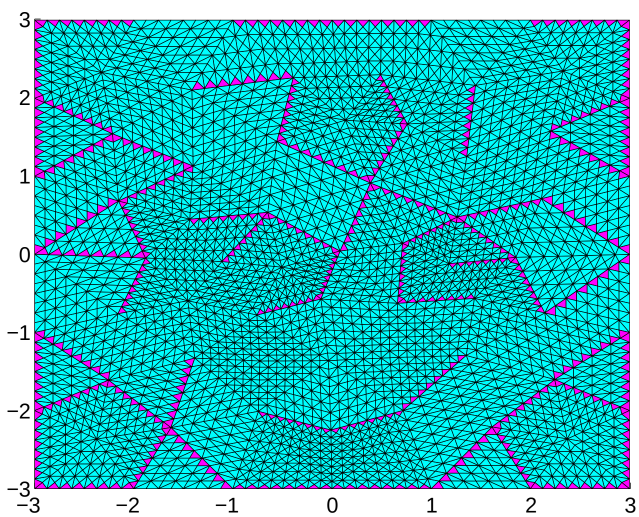


(b) Refinement step 1, 480 triangles, 356 terminal triangles (blue color)

Figure 18. Global 4T-LE refinement. Delaunay mesh

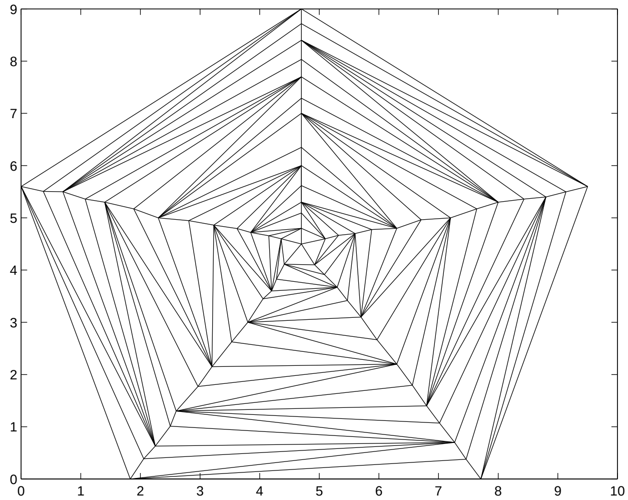


(a) Refinement step 2, 1920 triangles, 1672 terminal triangles (blue color)

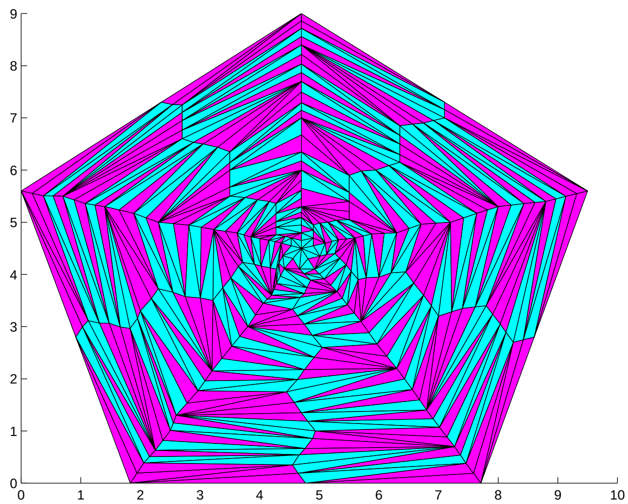


(b) Refinement step 3, 7680 triangles, 7184 terminal triangles (blue color)

Figure 19. Global 4T-LE refinement. Delaunay mesh

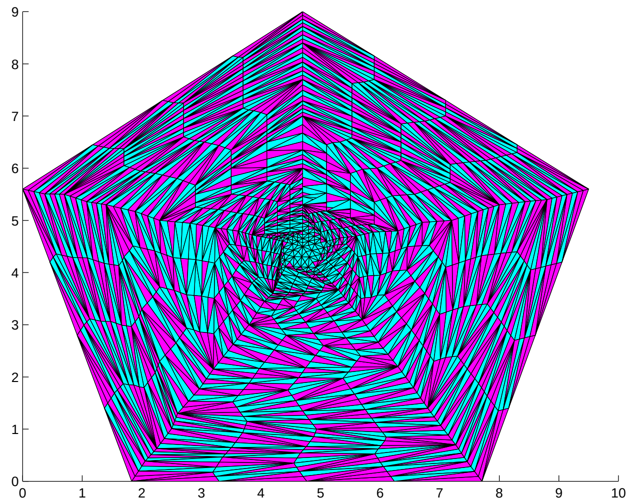


(a) Initial mesh, 0 terminal triangles, 125 triangles

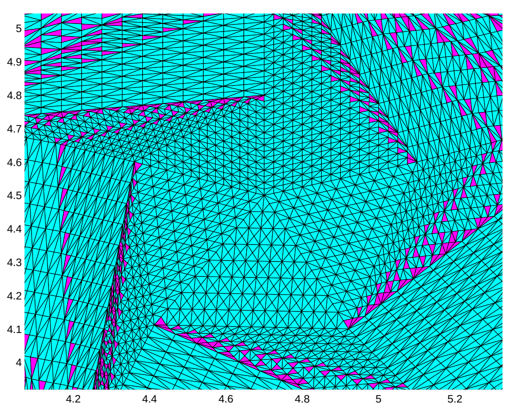


(b) Refinement step 1, 246 terminal triangles (blue color), 500 triangles

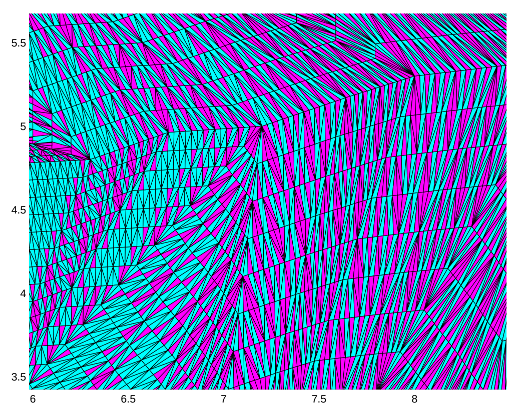
Figure 20. Global 4T-LE refinement. Pentagonal mesh



(a) Refinement step 2, 1088 terminal triangles (blue color), 2000 total triangles



(b) Refinement step 4, 20724 terminal triangles (blue color), 32000 total triangles. Interior zoom



(c) Refinement step 4, 20724 terminal triangles (blue color), 32000 total triangles. Exterior zoom

Figure 21. Global 4T-LE refinement. Pentagonal mesh

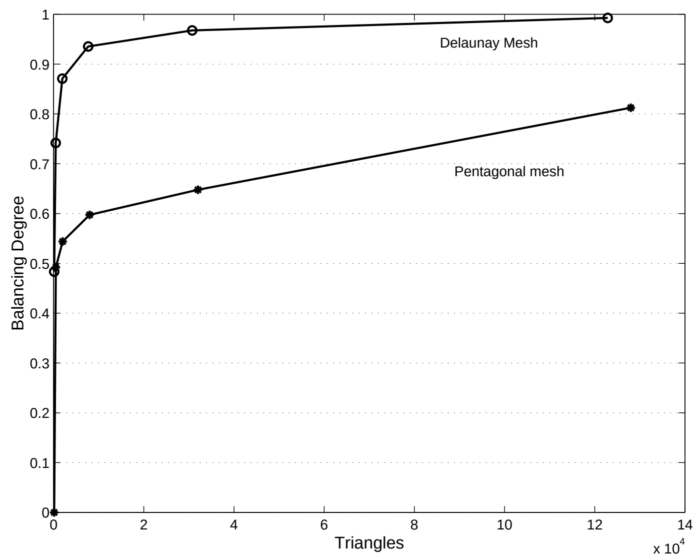


Figure 22. Balancing degree evolution. Delaunay and Pentagonal mesh

Table VII. Gran Canaria mesh. M1 and M2

Ref. Step	Triangles	M1 Mean	M2 Mean
0 (Initial mesh)	592	6.6199	3.4510
1	736	6.6902	3.5394
2	1230	6.5057	3.3837
3	2624	6.0191	2.9184
4	9258	5.5134	2.4265
5	30730	5.2478	2.3672
6	41448	5.2128	2.1891

5. CONCLUSIONS

In this work we have studied the propagation problem in longest edge based refinement algorithms in 2D. We proved both theoretically and empirically that the propagation of a single triangle refinement asymptotically extends to a few neighbor adjacent triangles. We found the limits of the propagation using the Conformity Neighborhood and two parameters

Table VIII. Gran Canaria mesh. Balancing degree

Ref. Step	Terminal triangles	Triangles	Balancing degree
0 (Initial mesh)	248	592	0.41891
1	326	736	0.44293
2	672	1230	0.54634
3	1588	2624	0.60518
4	7020	9258	0.75826
5	26282	30730	0.85525
6	35746	41448	0.86243

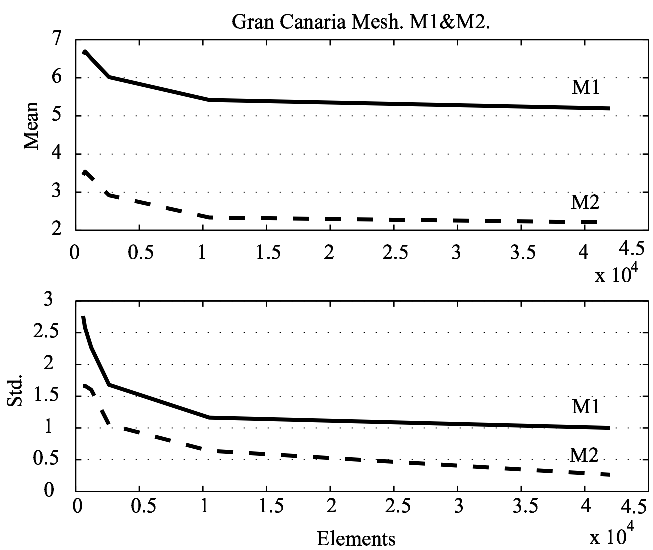
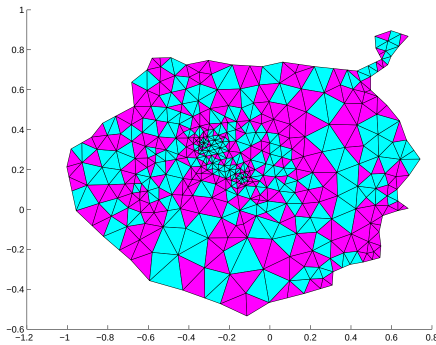
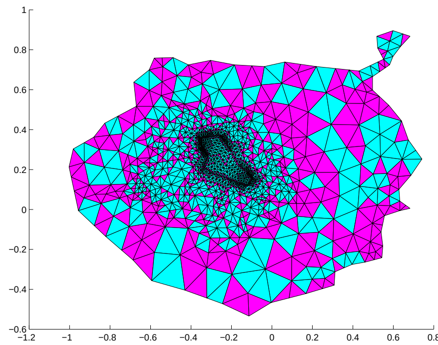


Figure 23. Gran Canaria mesh. Statistics of M1 and M2 evolution.

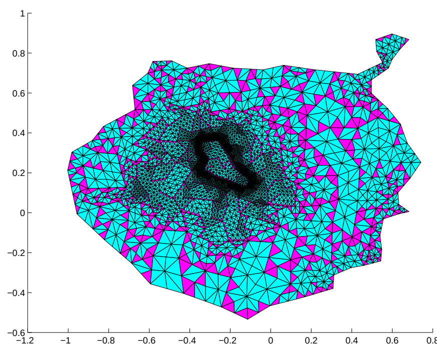
($M1(t)$ and $M2(t)$) which are related to the Longest-Edge Propagation Path (LEPP). When iterative (uniform or local) refinement is applied to an initial arbitrary triangular mesh, we found that the average of the parameters $M1(t)$ tends to 5 and the average of $M2(t)$ tends to 2. This guarantees for local refinement that the size of the triangulation is bounded and that the time cost of the algorithms is of constant complexity (Proposition 9). We also have introduced the concept of balancing degree (ratio between the terminal and total triangles in a mesh) for longest edge refinement of meshes and have proved that the Balancing Degree asymtotically



(a) Refinement step 1, 326 terminal triangles (blue color), 736 total triangles



(b) Refinement step 3, 1588 terminal triangles (blue color), 2624 total triangles. Interior zoom



(c) Refinement step 4, 7020 terminal triangles (blue color), 9258 total triangles.

Figure 24. Local 4T-LE refinement. Gran Canaria mesh

tends to 1. These results are also connected with the improvement of the generated meshes obtained by the *4T-LE* iterative refinement. Finally, the polyline and the Cantor set of a triangulation have been defined and some of their properties have been established. This concept could be of utility in order to improve the mesh by nodes movement [9], since the Cantor set points out the region of non terminal triangles and hence the improvement could turn these triangles to terminal triangles, but this idea requires further research.

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