

## 92.59 A recurrence relation for Fibonacci sums: a combinatorial approach

Consider the sums of Fibonacci numbers:  $\tau_n = \sum_{k=1}^n F_k$ , where  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$ , for  $n \geq 2$ . It is easy to prove that sequence  $\{\tau_n\}$  for  $n \geq 2$  satisfies the following recurrence relation [1]:

*Property 1:*

$$\tau_n = 1 + \tau_{n-1} + \tau_{n-2}. \quad (1)$$

Here, following the combinatorial interpretation for Fibonacci numbers used in [2, 3], we give a purely combinatorial proof for this recurrence relation.

A board of length  $n$  will be called here an  $n$ -board. We will consider such a board *tilled* with squares and dominoes, and in order to visually distinguish the squares from the dominoes we will consider white squares and black dominoes. Let us call these objects *black and white  $n$ -tilings*. Let  $f_n$  count the number of black and white  $n$ -tilings. For example,  $f_1 = 1$  since a length 1 board can only be covered by a white square;  $f_2 = 2$  since a board of length 2 can be covered with two white squares or one black domino. Similarly,  $f_3 = 3$  since a board of length 3 can be covered by 3 white squares or a black domino and a white square in one of 2 orders. We let  $f_0 = 1$  count the empty board. Then for  $n \geq 2$ ,  $f_n$  satisfies the Fibonacci recurrence

$$f_n = f_{n-1} + f_{n-2}$$

since a board of length  $n$  ends either in a white square preceded by a coloured  $(n-1)$ -tiling (tilled in  $f_{n-1}$  ways) or a black domino preceded by a coloured  $(n-2)$ -tiling (tilled in  $f_{n-2}$  ways). Since  $f_0 = 1 = F_1$  and  $f_1 = 1 = F_2$ , we see that for all  $n \geq 0$ ,  $f_n = F_{n+1}$ . Thus  $f_4 = 5$  enumerates the tilings:



FIGURE 1: All five black and white tilings of the 4-board

We use the following combinatorial interpretation for  $\tau_n = \sum_{k=1}^n F_k = \sum_{k=0}^{n-1} f_k$ :

**Lemma 1:**  $\tau_n$  gives the number of tilings of an  $(n+1)$ -board with at least one domino.

Note that we may obtain this result counting such tilings by the position of the last domino. See, for example, [3, Identity 1].

In order to prove relation (1), we count the number of black and white tilings of an  $(n+1)$ -board with at least one domino:

*Answer 1:* By the previous Lemma, there are  $\tau_n$  such tilings.

*Answer 2:* Consider the two different possibilities for the preceding cell to the last domino. If the preceding cell is covered by a white square, then for different positions of this last domino we get  $\tau_{n-1}$  tilings. On the other hand, if the preceding two cells correspond to another domino we get  $\tau_{n-2}$  tilings. Finally we have to count the tiling in which the last domino is precisely covering the two first cells of the  $(n+1)$ -board, as shown in Figure 2.

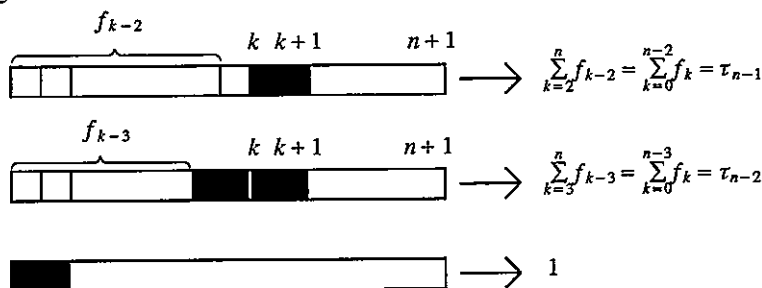


FIGURE 2: Possibilities for the last black domino in an  $(n+1)$ -board

### Generalisation

Pell numbers  $P_n$  are defined by  $P_0 = 0$ ,  $P_1 = 1$  and  $P_n = 2P_{n-1} + P_{n-2}$  for  $n \geq 2$ . (See [4], sequence [A000129], for more details about the Pell numbers). In order to provide combinatorial arguments for the Pell numbers, we will consider the sequence of numbers denoted  $p_n$  and defined as  $p_n = P_{n+1}$  for  $n \geq 0$ . The motivation for considering  $p_n$  is simple:  $p_n$  counts the number of tilings of a board of length  $n$  using white squares, black squares, and grey dominoes. Thus, for example,  $p_0 = 1$  counts the empty tiling;  $p_1 = 2$  because a board of length 1 can be covered (exactly) by either one white square or one black square. Similarly,  $p_2 = 5$  since a board of length 2 can be covered by two white squares or two black squares or one white square and one black square, in either order, or one grey domino.

Let us consider the sums of Pell numbers:

$$\sigma_n = \sum_{k=1}^n P_k = \sum_{k=0}^{n-1} p_k.$$

*Lemma 2:*  $\sigma_n$  gives the number of tilings of an  $(n+1)$ -board with at least one domino, in which the last domino is followed only by white squares.

It is easy to prove that sequence  $\{\sigma_n\}$  for  $n \geq 2$  satisfies the following recurrence relation:

*Property 2:*

$$\sigma_n = 1 + 2\sigma_{n-1} + \sigma_{n-2}. \quad (2)$$

To close this section, we consider a two-parameter generalisation of  $f_n$ , denoted  $f_n^{a,b}$  and defined by  $f_0^{a,b} = 1$ ,  $f_1^{a,b} = a$  and  $f_n^{a,b} = bf_{n-2}^{a,b} + f_{n-1}^{a,b}$  for  $n \geq 2$  where  $a$  and  $b$  are natural numbers. Thus  $f_n^{1,1} = f_n$ ,  $f_n^{2,1} = p_n$  and  $f_n^{a,b}$  counts the number of tilings of an  $n$ -board where  $a$  different colours of squares and  $b$  different colours of dominoes are available. (See, for example, [5]). For simplicity we will consider one of the square colours to be white, and one of the domino colours black.

Let us call  $\tau_n^{a,b} = \sum_{k=0}^{n-1} f_k^{a,b}$ . For this generalised Fibonacci sequence we get, in analogous way to before, the following results:

*Lemma 3:*  $\tau_n^{a,b}$  gives the number of tilings of an  $(n+1)$ -board with at least one black domino, in which the last black domino is followed only by white squares.

*Property 3:*

$$\tau_n^{a,b} = 1 + a\tau_{n-1}^{a,b} + b\tau_{n-2}^{a,b}. \quad (3)$$

### References

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