## 94.18 Proof without words: Two inequalities proved by convexity

The purpose of this note is to prove, almost without words, the following inequalities:

(1) if $\alpha > 1$, and $0 < \omega < 1$, then $\dfrac{(1 + \omega)^\alpha - 1}{(1 + \omega)^\alpha - \omega^\alpha} > \omega$;

(2) if $0 < \alpha < 1$, and $0 < \omega < 1$, then $\dfrac{(1 + \omega)^\alpha - 1}{(1 + \omega)^\alpha - \omega^\alpha} < \omega$.

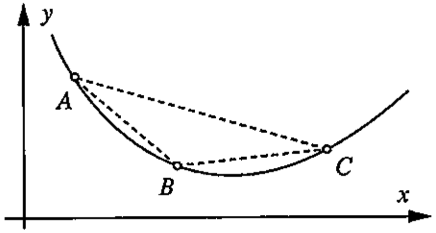


FIGURE 1: Convex function $f$ and three segments

The approach is by using convexity [1]. If $f$ is a convex function, such as the one illustrated in the figure above, then the slopes of the segments $AB$, $BC$ and $AC$ satisfy

$$m_{AB} < m_{AC} < m_{BC}.$$

As a consequence, many inequalities can be established by using different convex fuctions. For example, in order to prove Bernoulli's inequality

$$(1 + x)^n \geqslant 1 + nx, \text{ for all } n \in \mathbb{N}, x > -1$$

the convexity of $f(x) = x^n$ may be used, as the reader can check.

To prove (1), let $f(x) = x^\alpha$ for $\alpha > 1$, $A = (\omega, f(\omega))$, $B = (1, f(1))$ and $C = (1 + \omega, f(1 + \omega))$.

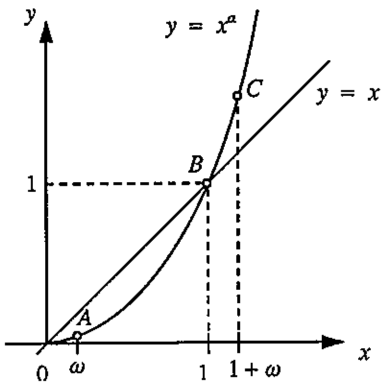


FIGURE 2: Function $y = x^\alpha$ for $\alpha > 1$



From $m_{AC} < m_{BC}$ in Figure 2, we have

$$\frac{(1 + \omega)^\alpha - \omega^\alpha}{1} < \frac{(1 + \omega)^\alpha - 1}{\omega}$$

from which (1) follows.

An analogous result for concave functions establishes inequality (2). This inequality is illustrated in Figure 3.

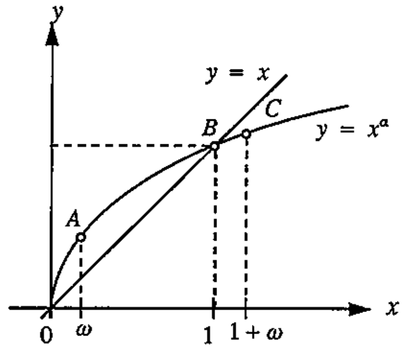


FIGURE 3: Function $y = x^\alpha$ for $0 < \alpha < 1$

From $m_{BC} < m_{AC}$ in Figure 3, we have

$$\frac{(1 + \omega)^\alpha - 1}{\omega} < \frac{(1 + \omega)^\alpha - \omega^\alpha}{1}$$

from which (2) follows.

*Reference*

1. A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

ÁNGEL PLAZA and SERGIO FALCÓN
*ULPGC, 35017-Las Palmas G.C., Spain*
e-mail: *aplaza@dmat.ulpgc.es*