1202. Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.

Let $p \geq 1$ be a natural number. Prove that

$$a) \quad \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} x^{pk} - \frac{1}{1-x^p}\right) = \frac{\ln(1-x^p)}{1-x^p}, \quad -1 < x < 1.$$

$$b) \quad \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} (-1)^k x^{pk} - \frac{1}{1+x^p}\right) = \frac{\ln(1+x^p)}{1+x^p}, \quad -1 < x < 1.$$

Solution: BY SERGIO FALCON AND ANGEL PLAZA, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain

It is enough to prove $a)$ since $b)$ follows from $a)$ by changing x^p by $-x^p$.

First we change variable $x^p = y$, with $-1 < y < 1$, since $p \geq 1$ is a natural number and $-1 < x < 1$. So, the left-hand side of $a)$ is now:

$$\begin{aligned}
\text{(LHS)} &= \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} y^k - \frac{1}{1-y}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} y^k - \sum_{k=0}^{\infty} y^k\right) \\
&= \sum_{n=1}^{\infty} \frac{-1}{n} \sum_{k=n}^{\infty} y^k = -\sum_{n=1}^{\infty} \frac{y^n}{n(1-y)}.
\end{aligned}$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{y^n}{n} = \sum_{n=1}^{\infty} \int_0^y t^{n-1}\, dt = \int_0^y \sum_{n=1}^{\infty} t^{n-1}\, dt = \int_0^y \frac{1}{1-t}\, dt = -\ln(1-y),$$

and then (LHS) $= -\displaystyle\sum_{n=1}^{\infty} \frac{y^n}{n(1-y)} = \frac{\ln(1-y)}{1-y}$, where by using again that $y = x^p$ the right-side of $a)$ is obtained.

Note that the interchange of the order of summation and integration is valid because $\displaystyle\sum_{n=1}^{\infty} \frac{y^n}{n} = -\ln(1-y)$ converges on $-1 \leq t < 1$, so the series converges uniformly on any closed interval in $(-1, 1)$. For $x = -1$, the desired equality follows from the Abel summation by parts formula. $\square$