A PROOF OF $\sin 2x = 2\sin x \cos x$

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Here we show that a trigonometric identity as $\sin 2x = 2\sin x\cos x$, may be proved ultimately by combinatorial arguments. The proof only uses the Taylor expansions of $\sin x$, $\cos x$, and the Cauchy product of two power series. That is,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Thus if we write

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\end{aligned}$$

by the Cauchy product, $\cos x \sin x$ can be written:

$$\cos x \sin x = \sum_{n=0}^{\infty} c_n x^n, \text{ where } c_n = \left\{ \begin{array}{ccc} \displaystyle\sum_{k=0}^{m} \frac{(-1)^k(-1)^{m-k}}{(2k+1)!(2m-2k)!} & \text{if} & n = 2m+1 \\ 0 & \text{if} & n = 2m \end{array} \right\}$$

Since

$$\frac{1}{2}\sin 2x = x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \frac{2^6x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n+1)!} x^{2n+1}$$

the only thing we have to prove is the identity

$$\sum_{k=0}^{m} \frac{1}{(2k+1)!(2m-2k)!} = \frac{2^{2m}}{(2m+1)!}$$

Or, equivalently,

$$\begin{aligned}\sum_{k=0}^{m} \frac{(2m+1)!}{(2k+1)!(2m-2k)!} &= 2^{2m} \\ \sum_{k=0}^{m} \binom{2m+1}{2k+1} &= 2^{2m}\end{aligned}$$

Date: November 12, 2009.

The last equation is a consequence of the facts:

$$(1+1)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} \quad = \quad 2^{2m+1} \tag{1}$$

$$(1-1)^{2m+1} = \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \quad = \quad 0 \tag{2}$$

By Eq. (2), $\sum_{k=0}^{m} \binom{2m+1}{2k+1} = \sum_{k=0}^{m} \binom{2m+1}{2k}$, and using Eq. (1), the last sums are both equal to 2^{2m} and the proof is done.

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