A PROOF OF $\sinh 2x = 2 \sinh x \cosh x$

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Here we show that a hyperbolic identity as $\sinh 2x = 2 \sinh x \cosh x$, may be proved ultimately by combinatorial arguments. The proof only uses the Taylor expansions of $\sinh x$, $\cosh x$, and the Cauchy product of two power series. That is,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Thus if we write

$$\begin{aligned}
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\end{aligned}$$

by the Cauchy product, $\sinh x \cosh x$ can be written:

$$\sinh x \cosh x = \sum_{n=0}^{\infty} c_n x^n, \text{ where } c_n = \left\{ \begin{array}{ccc} \displaystyle\sum_{k=0}^{m} \frac{1}{(2k+1)!(2m-2k)!} & \text{if} & n = 2m+1 \\ 0 & \text{if} & n = 2m \end{array} \right\}$$

Since

$$\frac{1}{2} \sinh 2x = x + \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} + \frac{2^6 x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

the only thing we have to prove is the identity

$$\sum_{k=0}^{m} \frac{1}{(2k+1)!(2m-2k)!} = \frac{2^{2m}}{(2m+1)!}$$

Or, equivalently,

$$\begin{aligned}
\sum_{k=0}^{m} \frac{(2m+1)!}{(2k+1)!(2m-2k)!} &= 2^{2m} \\
\sum_{k=0}^{m} \binom{2m+1}{2k+1} &= 2^{2m}
\end{aligned}$$

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The last equation is a consequence of the facts:

$$(1+1)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1} \tag{1}$$

$$(1-1)^{2m+1} = \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} = 0 \tag{2}$$

By Eq. (2), $\displaystyle\sum_{k=0}^{m} \binom{2m+1}{2k+1} = \sum_{k=0}^{m} \binom{2m+1}{2k}$, and using Eq. (1), the last sums are both equal to 2^{2m} and the proof is done.

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