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On k-Fibonacci numbers of arithmetic indexes

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ABSTRACT

In this paper, we study the sums of k-Fibonacci numbers with indexes in an arithmetic sequence, say $an+r$ for fixed integers a and r. This enables us to give in a straightforward way several formulas for the sums of such numbers.

1. Introduction

One of the more studied sequences is the Fibonacci sequence [1–3], and it has been generalized in many ways [4–10]. Here, we use the following one-parameter generalization of the Fibonacci sequence.

Definition 1. For any integer number $k \geqslant 1$, the kth Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by

$$F_{k,0}=0,\quad F_{k,1}=1,\quad \text{and}\quad F_{k,n+1}=kF_{k,n}+F_{k,n-1}\quad \text{for } n\geqslant 1.$$

Note that for $k=1$ the classical Fibonacci sequence is obtained while for $k=2$ we obtain the Pell sequence. Some of the properties that the k-Fibonacci numbers verify and that we will need later are summarized below [11–15]:

- [Binet's formula] $F_{k,n}=\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}$, where $\sigma_1=\frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2=\frac{k-\sqrt{k^2+4}}{2}$. These roots verify $\sigma_1+\sigma_2=k$, and $\sigma_1\cdot\sigma_2=-1$
- [Catalan's identity] $F_{k,n-r}F_{k,n+r}-F_{k,n}^2=(-1)^{n+1-r}F_{k,r}^2$
- [Simson's identity] $F_{k,n-1}F_{k,n+1}-F_{k,n}^2=(-1)^n$
- [D'Ocagne's identity] $F_{k,m}F_{k,n+1}-F_{k,m+1}F_{k,n}=(-1)^nF_{k,m-n}$
- [Convolution Product] $F_{k,n+m}=F_{k,n+1}F_{k,m}+F_{k,n}F_{k,m-1}$

In this paper, we study different sums of k-Fibonacci numbers. Sums of Fibonacci numbers appear in different contexts, even they are related with the dimensionality of heterotic superstrings [16,17]. We focus here on the subsequences of k-Fibonacci numbers with indexes in an arithmetic sequence, say $an+r$ for fixed integers a, r with $0\leqslant r\leqslant a-1$. Several formulas for the sums of such numbers are deduced in a straightforward way.

2. On the k-Fibonacci numbers of kind $an+r$

Let us prove two lemmas that we will need later.

Lemma 2. *For all integer n ($n\geqslant 1$):*

$$\sigma_1^n+\sigma_2^n=F_{k,n+1}+F_{k,n-1}. \tag{1}$$

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Proof. Applying Binet's formula and taking into account that $\sigma_1\sigma_2 = -1$

$$F_{k,n+1} + F_{k,n-1} = \frac{1}{\sigma_1 - \sigma_2}(\sigma_1^{n+1} - \sigma_2^{n+1} + \sigma_1^{n-1} - \sigma_2^{n-1}) = \frac{1}{\sigma_1 - \sigma_2}\left(\sigma_1^n\left(\sigma_1 + \frac{1}{\sigma_1}\right) - \sigma_2^n\left(\sigma_2 + \frac{1}{\sigma_2}\right)\right)$$
$$= \frac{1}{\sigma_1 - \sigma_2}(\sigma_1^n(\sigma_1 - \sigma_2) + \sigma_2^n(\sigma_1 - \sigma_2)) = \sigma_1^n + \sigma_2^n. \qquad \square$$

Lemma 3. $F_{k,a(n+2)+r} = (F_{k,a-1} + F_{k,a+1})F_{k,a(n+1)+r} - (-1)^a F_{k,an+r}$

Proof. Taking into account Lemma 2 and Binet's formula:

$$(F_{k,a-1} + F_{k,a+1})F_{k,a(n+1)+r} = (\sigma_1^a + \sigma_2^a)\frac{\sigma_1^{a(n+1)+r} - \sigma_2^{a(n+1)+r}}{\sigma_1 - \sigma_2} = \frac{1}{\sigma_1 - \sigma_2}(\sigma_1^{a(n+2)+r} - \sigma_2^{a(n+2)+r} + (-1)^a\sigma_1^{an+r} - (-1)^a\sigma_2^{an+r})$$
$$= F_{k,a(n+2)+r} + (-1)^a F_{k,an+r}. \qquad \square$$

Let us denote $F_{k,n-1} + F_{k,n+1}$ by $L_{k,n}$ (numbers $L_{k,n}$ are called k-Lucas numbers).

Then previous formula becomes

$$F_{k,a(n+2)+r} = L_{k,a}F_{k,a(n+1)+r} - (-1)^a F_{k,an+r}. \tag{2}$$

Eq. (2) gives the general term of the k-Fibonacci sequence $\{F_{k,an+r}\}_{n=0}^{\infty}$ as a linear combination of the two preceding terms. Note that, applying iteratively this formula, the general term can be written as a non-linear combination of the two first terms of the sequence:

$$F_{k,an+r} = \left(\sum_{i=0}^{[\frac{n-1}{2}]}(-1)^{(a+1)i}L_{k,a}^{n-1-2i}\binom{n-1-i}{i}\right)F_{k,a+r}^{n-1-2i} + \left(\sum_{i=0}^{[\frac{n-2}{2}]}(-1)^{(a+1)(i+1)}L_{k,a}^{n-2-2i}\binom{n-2-i}{i}\right)F_{k,r}^{n-2-i}.$$

In this way, the general term of sequence $\{F_{k,an+r}\}$ is written in function of the two first terms. In particular, for $a = 1$ it is $r = 0$, see [12], we have

$$F_{k,n} = k^{n-1-2i}\sum_{i=0}^{[\frac{n-1}{2}]}\binom{n-1-i}{i}.$$

2.1. Generating function of the sequence $\{F_{k,an+r}\}$

Let $f_{a,r}(k,x)$ be the generating function of the sequence $\{F_{k,an+r}\}$, with $0 \leqslant r \leqslant a-1$. That is, $f_{a,r}(k,x) = F_{k,r} + F_{k,a+r}x + F_{k,2a+r}x^2 + \cdots$. After some easy algebra

$$(1 - L_{k,a}x + (-1)^a x^2)f_{a,r}(k,x) = F_{k,r} + (F_{k,a+r} - F_{k,r}L_{k,a})x + \sum_{n\geqslant 2}(F_{k,a(n+2)+r} - L_{k,a}F_{k,a(n+1)+r} + (-1)^a F_{k,an+r})x^n.$$

First, taking into account Lemma 3, the series of the Right Hand Side vanishes.

On the other hand, the Convolution Product Identity establishes that $F_{k,r+a} = F_{k,r}F_{k,a+1} + F_{k,r-1}F_{k,a}$, so $F_{k,a+r} - F_{k,r}L_{k,a} = F_{k,a}F_{k,r+1} - F_{k,a+1}F_{k,r}$.

Finally, $F_{k,a-r} = F_{k,-r}F_{k,a+1} + F_{k,-r-1}F_{k,a} = (-1)^r(-F_{k,a+1}F_{k,r} + F_{k,a}F_{k,r+1})$, and the generating function for the initial power series is

$$f_{a,r}(k,x) = \frac{F_{k,r} + (-1)^r F_{k,a-r}x}{1 - L_{k,a}x + (-1)^a x^2}. \tag{3}$$

2.1.1. Particular cases

The generating functions of sequences $\{F_{k,an+r}\}$ for different values of parameters a and r are

(1) $a = 1$ and then $r = 0$: $f_{1,0}(k,x) = \frac{x}{1-kx-x^2}$ [12,15]
(2) $a = 2$:
  (a) $r = 0$: $f_{2,0}(k,x) = \frac{kx}{1-(k^2+2)x+x^2}$
  (b) $r = 1$: $f_{2,1}(k,x) = \frac{1-x}{1-(k^2+2)x+x^2}$

(3) $a = 3$:
   (a) $r = 0$: $f_{3,0}(k,x) = \frac{(k^2+1)x}{1-(k^3+3k)x-x^2}$
   (b) $r = 1$: $f_{3,1}(k,x) = \frac{1-kx}{1-(k^3+3k)x-x^2}$
   (c) $r = 2$: $f_{3,2}(k,x) = \frac{k+x}{1-(k^3+3k)x-x^2}$

2.2. Sum of k-Fibonacci numbers of kind $an + r$

In this section, we study the sum of the k-Fibonacci numbers of kind $an + r$, with a an integer number, and $r = 0, 1, 2, \ldots, a-1$.

Theorem 4. *Sum of the k-Fibonacci numbers of kind $an + r$*

$$\sum_{i=0}^{n} F_{k,ai+r} = \frac{F_{k,a(n+1)+r} - (-1)^a F_{k,an+r} - F_{k,r} - (-1)^r F_{k,a-r}}{F_{k,a+1} + F_{k,a-1} - (-1)^a - 1}. \tag{4}$$

Proof. Applying Binnet's formula to $S_{k,an+r} = \sum_{i=0}^{n} F_{k,ai+r}$, we get

$$\begin{aligned} S_{k,an+r} &= \sum_{i=0}^{n} \frac{\sigma_1^{ai+r} - \sigma_2^{ai+r}}{\sigma_1 - \sigma_2} = \frac{1}{\sigma_1 - \sigma_2}\left(\sum_{i=0}^{n} \sigma_1^{ai+r} - \sum_{i=0}^{n} \sigma_2^{ai+r}\right) = \frac{1}{\sigma_1 - \sigma_2}\left(\frac{\sigma_1^{an+r+a} - \sigma_1^r}{\sigma_1^a - 1} - \frac{\sigma_2^{an+r+a} - \sigma_2^r}{\sigma_2^a - 1}\right) \\ &= \frac{1}{(\sigma_1\sigma_2)^a - \sigma_1^a - \sigma_2^a + 1} \frac{1}{\sigma_1 - \sigma_2}\left(\sigma_1^{an+r}(\sigma_1\sigma_2)^a - \sigma_1^r\sigma_2^a - \sigma_1^{a(n+1)+r} + \sigma_1^r - \sigma_2^{an+r}(\sigma_1\sigma_2)^a + \sigma_1^a\sigma_2^r + \sigma_2^{a(n+1)+r} - \sigma_2^r\right) \\ &= \frac{1}{(-1)^a - (\sigma_1^a + \sigma_2^a) + 1}\left((-1)^a \frac{\sigma_1^{an+r} - \sigma_2^{an+r}}{\sigma_1 - \sigma_2} - \frac{\sigma_1^{a(n+1)+r} - \sigma_2^{a(n+1)+r}}{\sigma_1 - \sigma_2} + \frac{\sigma_1^r - \sigma_2^r}{\sigma_1 - \sigma_2} + \frac{-\sigma_2^a(\sigma_1^a(-\sigma_1)^{-r} - \sigma_2)^{-r}}{\sigma_1 - \sigma_2}\right) \\ &= \frac{F_{k,a(n+1)+r} - (-1)^a F_{k,an+r} - F_{k,r} - (-1)^r F_{k,a-r}}{F_{k,a+1} + F_{k,a-1} - (-1)^a - 1}, \end{aligned}$$

where we have used Eq. (2). □

For $k = 1, 2, 3$ different sequences of these partial sums are listed in OEIS [18].

Corollary 5. *Sum of odd k-Fibonacci numbers*
If $a = 2p + 1$ then Eq. (4) *is*

$$\sum_{i=0}^{n} F_{k,(2p+1)i+r} = \frac{F_{k,(2p+1)(n+1)+r} + F_{k,(2p+1)n+r} - F_{k,r} - (-1)^r F_{k,(2p+1)-r}}{F_{k,2p+2} + F_{k,2p}}. \tag{5}$$

For example

(1) If $p = 0$ then $a = 1 \to r = 0$, and $\sum_{i=0}^{n} F_{k,i} = \frac{F_{k,n+1} + F_{k,n} - F_{k,0} - F_{k,1}}{F_{k,2} + F_{k,0}} = \frac{F_{k,n+1} + F_{k,n} - 1}{k}$ [11,12]
   (a) For $k = 1$, for the classical Fibonacci sequence it is

$$\sum_{i=0}^{n} F_i = \frac{F_{n+1} + F_n - 1}{k} = F_{n+2} - 1.$$

   (b) For $k = 2$, for the Pell sequence we obtain $\sum_{i=0}^{n} P_i = \frac{P_{n+1} + P_n - 1}{2}$
(2) If $p = 1 \to a = 3$, then $\sum_{i=0}^{n} F_{k,3i+r} = \frac{F_{k,3(n+1)+r} + F_{k,3n+r} - F_{k,r} - (-1)^r F_{k,3-r}}{k^3+3k}$
   (a) $r = 0$: $\sum_{i=0}^{n} F_{k,3i} = \frac{F_{k,3n+3} + F_{k,3n} - k^2 - 1}{k^3+3k}$
   For the classical Fibonacci sequence, $k = 1$, it is

$$\sum_{i=0}^{n} F_{3i} = \frac{F_{3n+3} + F_{3n} - 2}{4}.$$

   (b) $r = 1$: $\sum_{i=0}^{n} F_{k,3i+1} = \frac{F_{k,3n+4} + F_{k,3n+1} + k - 1}{k^3+3k}$
   For the classical Fibonacci sequence, $k = 1$, it is

$$\sum_{i=0}^{n} F_{3i+1} = \frac{F_{3n+4} + F_{3n+1}}{4}.$$

(c) $r = 2$: $\sum_{i=0}^{n} F_{k,3i+2} = \frac{F_{k,3n+5}+F_{k,3n+2}-k-1}{k^3+3k}$
For the classical Fibonacci sequence, $k = 1$, it is

$$\sum_{i=0}^{n} F_{3i+2} = \frac{F_{3n+5} + F_{3n+2} - 2}{4}$$

(3) If $p = 2 \rightarrow a = 5$, then

$$\sum_{i=0}^{n} F_{k,5i+r} = \frac{F_{k,5(n+1)+r} + F_{k,5n+r} - F_{k,r} - (-1)^r F_{k,5-r}}{k^5 + 5k^3 + 5k}.$$

(a) $r = 0$: $\sum_{i=0}^{n} F_{k,5i} = \frac{F_{k,5n+5}+F_{k,5n}-k^4-3k^2-1}{k^5+5k^3+5k}$

(b) $r = 1$: $\sum_{i=0}^{n} F_{k,5i+1} = \frac{F_{k,5n+6}+F_{k,5n+1}+k^3+2k-1}{k^5+5k^3+5k}$

(c) $r = 2$: $\sum_{i=0}^{n} F_{k,5i+2} = \frac{F_{k,5n+7}+F_{k,5n+2}-k^2-k-1}{k^5+5k^3+5k}$

(d) $r = 3$: $\sum_{i=0}^{n} F_{k,5i+3} = \frac{F_{k,5n+8}+F_{k,5n+3}-k^2+k-1}{k^5+5k^3+5k}$

(e) $r = 4$: $\sum_{i=0}^{n} F_{k,5i+4} = \frac{F_{k,5n+9}+F_{k,5n+4}-k^3-2k-1}{k^5+5k^3+5k}$

Corollary 6. *Sum of even k-Fibonacci numbers*
If $a = 2p$ then Eq. *(4) is*

$$\sum_{i=0}^{n} F_{k,2pi+r} = \frac{F_{k,2p(n+1)+r} - F_{k,2pn+r} - F_{k,r} - (-1)^r F_{k,2p-r}}{F_{k,2p+1} + F_{k,2p-1} - 2}. \quad \square \qquad (6)$$

For example,

(1) If $p = 1 \rightarrow a = 2$, then

$$\sum_{i=0}^{n} F_{k,2i+r} = \frac{kF_{k,2n+1+r} - F_{k,r} - (-1)^r F_{k,2-r}}{k^2}.$$

(a) $r = 0$: $\sum_{i=0}^{n} F_{k,2i} = \frac{F_{k,2n+1}-1}{k}$For the classical Fibonacci sequence, $k = 1$, it is

$$\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1.$$

(b) $r = 1$: $\sum_{i=0}^{n} F_{k,2i+1} = \frac{F_{k,2n+2}}{k}$
For the classical Fibonacci sequence, $k = 1$, it is

$$\sum_{i=0}^{n} F_{2i+1} = F_{2n+2}.$$

(2) If $p = 2 \rightarrow a = 4$, then

$$\sum_{i=0}^{n} F_{k,4i+r} = \frac{F_{k,4(n+1)+r} - F_{k,4n+r} - F_{k,r} - (-1)^r F_{k,4-r}}{k^4 + 4k^2}.$$

(a) $r = 0$: $\sum_{i=0}^{n} F_{k,4i} = \frac{F_{k,4n+4}-F_{k,4n}-k^3-2k}{k^4+4k^2}$

(b) $r = 1$: $\sum_{i=0}^{n} F_{k,4i+1} = \frac{F_{k,4n+5}-F_{k,4n+1}+k^2}{k^4+4k^2}$

(c) $r = 2$: $\sum_{i=0}^{n} F_{k,4i+2} = \frac{F_{k,4n+6}-F_{k,4n+2}-2k}{k^4+4k^2}$

(d) $r = 3$: $\sum_{i=0}^{n} F_{k,4i+3} = \frac{F_{k,4n+7}-F_{k,4n+3}+k^2}{k^4+4k^2}$

2.3. Recurrence law for the sequence of sums of k-Fibonacci numbers of arithmetic indexes

It is relatively easy to prove by induction that the sequence $\{S_{k,an+r}\} = \{\sum_{i=0}^{n} F_{k,ai+r}\}$, verifies the recurrence relation $S_{k,a(n+1)+r} = L_{k,a}S_{k,an+r} + (-1)^{a+1}S_{k,a(n-1)+r} + F_{k,r} + (-1)^r F_{k,a-r}$.

$$\begin{aligned} S_{k,an+r} &= \sum_{i=0}^{n} F_{k,ai+r} = F_{k,r} + F_{k,a+r} + \sum_{i=2}^{n} L_{k,a}F_{k,a(i-1)+r} - (-1)^a F_{k,a(i-2)+r} = F_{k,r} + F_{k,a+r} + L_{k,a}\sum_{i=1}^{n-1} + F_{k,ai+r} - (-1)^a \sum_{i=0}^{n-2} + F_{k,ai+r} \\ &= F_{k,r} + F_{k,a+r} + L_{k,a}(S_{k,a(n-1)+r} - F_{k,r}) - (-1)^a S_{k,a(n-2)+r}. \end{aligned}$$

Now considering $S_{k,an+r}$ and $S_{k,a(n+1)+r}$:

$$\begin{aligned} S_{k,an+r} &= (1 - L_{k,a})F_{k,r} + F_{k,a+r} + L_{k,a}S_{k,a(n-1)+r} - (-1)^a S_{k,a(n-2)+r}, \\ S_{k,a(n+1)+r} &= (1 - L_{k,a})F_{k,r} + F_{k,a+r} + L_{k,a}S_{k,an+r} - (-1)^a S_{k,a(n-1)+r} \end{aligned}$$

by eliminating the terms $(1 - L_{k,a})F_{k,r} + F_{k,a+r}$, it is deduced:

$$S_{k,a(n+1)+r} = (1 + L_{k,a})S_{k,an+r} - (L_{k,a} + (-1)^a)S_{k,a(n-1)+r} + (-1)^a S_{k,a(n-2)+r}.$$

So, the characteristic polynomial of sequence $\{S_{k,an+r}\}$ is $r^3 = (1 + L_{k,a})r^2 - (L_{k,a} + (-1)^a)r + (-1)^a S_{k,a(n-2)+r}$, with roots $r_0 = 1$, $r_1 = \frac{L_{k,a}+\sqrt{L_{k,a}^2-4(-1)^a}}{2}$, and $r_2 = \frac{L_{k,a}-\sqrt{L_{k,a}^2-4(-1)^a}}{2}$. These numbers verify $r_1 + r_2 = L_{k,a}$, $r_1 \cdot r_2 = (-1)^a$, and $r_1 - r_2 = \sqrt{L_{k,a}^2 - 4(-1)^a}$. Then, the solution for $S_{k,an+r}$ is of the form $S_{k,an+r} = C_0 + C_1 r_1^n + C_2 r_2^n$. Having in mind the relations between r_1 and r_2, after some algebra is obtained

$$\begin{aligned} C_0 &= \frac{(r_1 - L_{k,a})F_{k,a+r} + (L_{k,a} - 1)F_{k,r}}{L_{k,a} - 1 - (-1)^a}, \\ C_1 &= \frac{(r_1 - L_{k,a})F_{k,a+r} + (-1)^a F_{k,r}}{(r_1 - r_2)(r_2 - 1)}, \\ C_2 &= -\frac{(r_2 - L_{k,a})F_{k,a+r} + (-1)^a F_{k,r}}{(r_1 - r_2)(r_1 - 1)}. \end{aligned}$$

Observe that, in the case $a = 1$, $r = 0$, and then the recurrence becomes $S_{n+1} = (1 + k)S_n - (k - 1)S_{n-1} - S_{n-2}$, which, for the classical Fibonacci (that is $k = 1$) reports $S_{n+1} = 2S_n - S_{n-2}$.

Let us now consider the alternating sequence $\{(-1)^n F_{k,an+r}\}$. In a similar way that in the preceding case, we can find that the generating function for this alternating sequence is $g_{a,r}(k,x) = \frac{F_{k,r}-(-1)^r F_{k,a-r}x}{1-L_{k,a}x+(-1)^a x^2}$. Moreover, the following result is given:

Theorem 7. *Alternating sum of the k-Fibonacci numbers of order $an + r$*

$$\sum_{i=0}^{n} (-1)^i F_{k,ai+r} = \frac{(-1)^n F_{k,a(n+1)+r} + (-1)^{n+a} F_{k,an+r} + (-1)^{r+1} F_{k,a-r} + F_{k,r}}{F_{k,a+1} + F_{k,a-1} + (-1)^a + 1}$$

which for different values of a and r reads as

(1) $\sum_{i=0}^{n} (-1)^i F_{k,i} = \frac{(-1)^n F_{k,n+1} - (-1)^n F_{k,n} - 1}{k}$
(2) $\sum_{i=0}^{n} (-1)^i F_{k,2i} = \frac{(-1)^n F_{k,2n+2} + (-1)^n F_{k,2n} - k}{k^2+4} = (-1)^n F_{k,n} F_{k,n+1}$
(3) $\sum_{i=0}^{n} (-1)^i F_{k,2i+1} = (-1)^n F_{k,n+1}^2$
(4) $\sum_{i=0}^{n} (-1)^i F_{k,4i} = \frac{(-1)^n F_{k,4n+2} - k}{k^2+2}$
(5) $\sum_{i=0}^{n} (-1)^i F_{k,4i+1} = \frac{(-1)^n F_{k,4n+3} + 1}{k^2+2}$
(6) $\sum_{i=0}^{n} (-1)^i F_{k,4i+2} = \frac{(-1)^n F_{k,4n+4}}{k^2+2}$
(7) $\sum_{i=0}^{n} (-1)^i F_{k,4i+3} = \frac{(-1)^n F_{k,4n+5} + 1}{k^2+2}$

3. Conclusions

We have studied the subsequences of k-Fibonacci numbers with indexes in an arithmetic sequence. In a compact and direct way many formulas for the sums of such numbers have been deduced.

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