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A new proof of the degeneracy property of the longest-edge n-section refinement scheme for triangular meshes

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ABSTRACT

In this note, by using complex variable functions, we present a new simpler proof of the degeneracy property of the longest-edge n-section of triangles for $n \geqslant 4$. This means that the longest-edge n-section of triangles for $n \geqslant 4$ produces a sequence of triangles with minimum interior angle converging to zero.

1. Introduction

Unstructured mesh generation and adaptive mesh refinement methods for two- and three-dimensional complex domains are very successful tools for the efficient solution of numerical application problems. A major drawback of these methods is that they may produce poorly shaped elements causing the numerical solution to be less accurate and more difficult to compute [1,9]. In the area of adaptive finite element methods, mesh refinement algorithms that maintain the *non-degeneracy* of the elements are certainly desirable. Non-degeneracy means that the minimum angle of the triangles is bounded away from zero when the partition or the refinement is applied.

The longest-edge bisection is one possible way of subdividing a triangle and constitutes an efficient scheme for reducing the obtuse angles in mesh refinement [3,6]. The non-degeneracy property of longest-edge bisection of triangles is formulated usually as follows: Let t be an initial triangle with smallest angle α. Then the longest-edge partition of triangle t and its descendants only produces triangles whose smallest angles are always greater or equal to $\alpha/2$ [7]. For the case of longest-edge trisection of triangles a similar bound has been recently found using a space of triangular shapes endowed with a hyperbolic metric. In fact, for this case the smallest angle produced by longest-edge trisection is always greater or equal to α/c with $c = \frac{\pi/3}{\arctan(\sqrt{3}/11)}$ [5,2].

2. Longest-edge n-section of the triangles, for $n \geqslant 4$

In a recent paper [8], Suárez et al. have proved by elementary geometric arguments the degeneracy property of the triangular meshes obtained by iterative application of longest-edge n-section of the triangles, when $n \geqslant 4$. In this short note, we propose a new simpler proof of the same property by using a normalization process and elementary complex variable functions.

Theorem 1 [8]. *The iterative application of longest-edge n-section when $n \geqslant 4$ to an initial triangle ΔABC generates a sequence of new triangles in which* $\lim_{k\to\infty}\alpha_k = 0$, α_k *being the minimum triangle angle in iteration k.*

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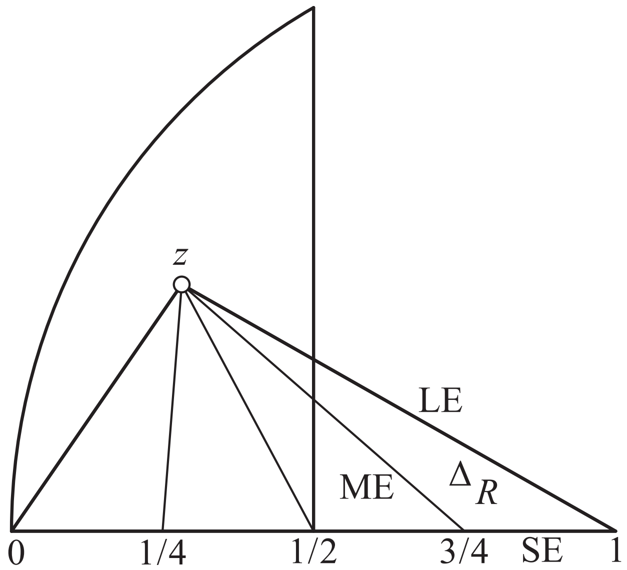

Fig. 1. Normalized shape region $\mathfrak{S}$ and longest-edge 4-section of triangle determined by z: LE, ME and SE are respectively longest, medium and shortest edge of triangle Δ_R.

Proof. A geometrical diagram to study the shape of triangles generated by iterative application of triangle subdivision has been introduced in [4]. The map diagram provides a convenient way to visualize the evolution and migration of element shapes. We consider all the triangles normalized in such a way that for a given triangle the longest-edge is scaled to have unit length, with extreme points (0,0) and (1,0), and the shortest angle at point (1,0). It is clear, that after some Euclidean movements any triangle may be transformed in a similar triangle in that way.

Note that a normalized triangle is determined by the opposite point to the longest-edge. This point z will be called apex of the triangle and may be considered as a complex point. The region of the plane for the apexes of normalized triangles will be called *normalized shape region* and denoted by $\mathfrak{S}$:

$$\mathfrak{S} = \{z \in \mathbb{C},\ \mathrm{Im}(z) > 0,\ \mathrm{Re}(z) \leqslant 1/2,\ |z-1| \leqslant 1\}.$$

In this form, all similar triangles are represented once normalized by a unique point z in the normalized shape region. Note that then the shortest angle α is written as function of apex point z as $\alpha = \mathrm{Arg}(1-\bar{z})$.

Fig. 1 shows the normalized shape region $\mathfrak{S}$, and a triangle divided by the longest-edge 4-section. The sub-triangle with vertex at point (1,0) is denoted by Δ_R in Fig. 1. The shortest edge of Δ_R, SE in the figure, has extreme points (3/4,0) and (1,0). In general for the case of longest-edge n-section, being $n \geqslant 4$, the shortest edge of the sub-triangle with vertex point (1,0) would be $1/n$ length.

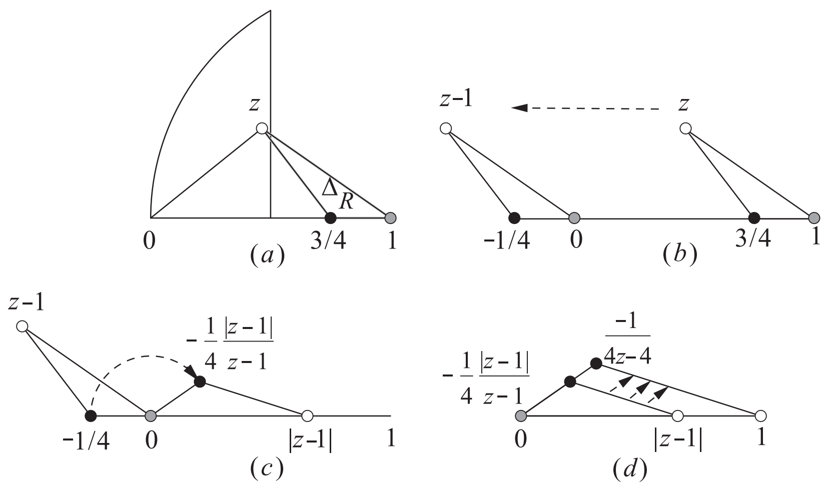

Fig. 2. Normalization of Triangle Δ_R for LE 4-section, expressed by compositions of complex variable functions.

Note that triangle Δ_R depends on the initial triangle, and so on point z. Therefore the normalization process of triangle Δ_R may be understood as a complex variable function $w(z)$ being z the apex point of the initial triangle and $w(z)$ the apex point of the triangle Δ_R once normalized. For the case $n = 4$, function $w(z)$ is obtained in Fig. 2, where $w(z) = \frac{-1}{4(z-1)}$. For the general case, $n \geqslant 4$, it would be $w(z) = \frac{-1}{n(z-1)}$.

Note that since $|z-1| \geqslant \frac{1}{2}$, then $\frac{\mathrm{Im} w(z)}{\mathrm{Im} z} \leqslant \frac{2}{n}$:

$$\frac{\mathrm{Im} w(z)}{\mathrm{Im} z} = \frac{\mathrm{Im} \frac{-1}{n(z-1)}}{\mathrm{Im} z} = \frac{\mathrm{Im} \frac{-((\bar{z}-1))}{n(|z-1|)}}{\mathrm{Im} z} \leqslant \frac{2}{n} \frac{\mathrm{Im}(1-\bar{z})}{\mathrm{Im} z} = \frac{2}{n}.$$

Therefore, after k compositions of functions w we get $\mathrm{Im} w^k(z) \leqslant \left(\frac{2}{n}\right)^k \mathrm{Im} z$. In other words, $\mathrm{Im} w^k(z)$ tends to zero as k goes to infinity, and so the argument of $1 - \overline{w^k(z)}$. This completes de proof. □

Note that, since the estimate $\frac{\mathrm{Im} w(z)}{\mathrm{Im} z} \leqslant \frac{2}{n}$ does not depend on point z, it also holds, for example, if z belongs to the boundary of the normalized region $\mathfrak{S}$. For instance, let $z = \frac{1}{2} + hi$ then the minimum initial angle is $\alpha = \arctan 2h$ while the minimum angle of $w(z)$ is $\alpha_{w(z)} = \arctan \frac{h}{\frac{n-2}{4} + nh^2}$ which tends to zero as h tends to zero.

3. Final remarks

Here the degeneracy property of the triangular partitions obtained by longest-edge n-section, for $n \geqslant 4$, of triangles has been proved. It has been proved that if α_k is the minimum angle of the triangles obtained by the LE-n-section of an initial triangle t, then α_k tends to zero for $k \to \infty$. This fact is in contrast with the non-degeneracy property of the longest-edge n-section for $n = 2, 3$.

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