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A comparative study between some bisection based partitions in 3D

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Abstract

In the last years many different strategies have been developed to locally refine triangular and tetrahedral meshes. In three dimensions, many algorithms can be seen as a generalization of their respective counterparts in two dimensions.

In this paper we study and compare three of the main algorithms used in the literature for the refinement of tetrahedral grids: Rivara–Levin, Plaza–Carey and Liu–Joe. We find out whether they are equivalent or not, and in what conditions the quality of the tetrahedra is also compared. We present numerical examples that support our study and that provide to engineers with practical notes for their numerical applications.

Keywords: Mesh; Refinement; Adaptivity; Tetrahedra

1. Introduction

Meshing algorithms are basic tools in any computer graphics and CAD software and play a significant role in many others fields such as geometric and engineering design, geometric modeling and finite element methods. Many algorithms have been developed in two dimensions, for example, the 4T algorithm of Rivara [16] is based on a edge bisection: First the longest edge is bisected, and the mid-edge point is

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connected to the vertex opposite, and then the newly formed vertex is used to subdivide the initial triangle in four. In the "newest vertex bisection" introduced by Mitchell in [10], the triangle edge to be bisected is identified without any computation.

In three dimensions there are two main approaches for subdividing a single tetrahedron: octasection and bisection. Octasection methods simultaneously create eight descendants for each tetrahedron. Methods based on bisection can also be devised easily to subdivide each tetrahedron in eight, but the primary stage consists in bisecting the tetrahedron in two. In this paper we focus on bisection edge based partition in 3D. Bänsch [1] presents an algorithm based on the selection of an edge as a *global refinement edge* in each tetrahedron, but imposes small perturbations of the coordinates of the nodes to avoid incompatibilities.

In [17] Rivara and Levin considered a pure three-dimensional longest-edge refinement method. Empirical experimentation was provided showing that the solid angle decreases slowly with the refinement iteration, and that a quality-element improvement behavior, holds in practice, as in the two-dimensional case. However, there have not been mathematical results guaranteeing the non-degeneracy property of the 3-dimensional mesh.

Liu and Joe [6,7] present an algorithm (*QLRB*) similar to that of Bänsch. They classify the tetrahedra in four types and set up the types of edges depending on the type of tetrahedron. The bisection edges are chosen without computation. This order depends on the assignation of types to the new edges. In this sense it can be said that the *QLRB* algorithm can be seen as a generalization to 3D of the Michell algorithm. In addition, a shape measure is introduced. The number of similarity classes is proved to be bounded and therefore the meshes cannot degenerate.

Several similar others studies are also reported in the literature. For instance, a recursive approach is proposed by Kossaczký [5]. This algorithm imposes certain restrictions and preprocessing in the initial mesh. The 3D algorithm is equivalent to that given in [1]. Maubach [8] develops an algorithm for n-simplicial grids generated by reflection. Although the algorithm is valid in any dimension and the number of similarity classes is bounded, it cannot be applied for a general tetrahedral grid. An additional closure refinement is needed to avoid incompatibilities. Recently, Mukherjee [11] has presented an algorithm equivalent to [1,6], and proves the equivalence with [8].

The algorithm developed by Plaza and Carey [14] is also based on bisection, but the point of view is different to those cited above since the three-dimensional approach is based on the two-dimensional one applied to the skeleton of the triangulation. The two-dimensional version is equivalent to the 4T Rivara algorithm, and the 3D one is the generalization to three dimensions of the 4T algorithm. The algorithm can be applied to any valid initial mesh without any restriction on the shape of the tetrahedra, since it is based on a previous classification of the edges based on their length.

The paper is organized as follows: First we introduce the main partitions on the plane and in the space. In the third section a comparative study is done between three different algorithms and finally, some conclusions about this work are summarized in the last section.

2. Basic definitions and preliminaries

A major class of refinement methods is based on the simplex bisection. Two of the most used bisection-based partitions in two dimensions are the pure longest-edge bisection and the 4-triangle longest-edge bisection.

Definition 1 (*Edge bisection and longest-edge bisection*). The longest-edge bisection of a triangle is the bisection of t by the midpoint of the longest-edge and its opposite vertex. If the chosen midpoint is not the longest-edge midpoint then it is said that a simple bisection has been performed.

Definition 2 (*The* 4*-triangles longest-edge* (4*T-LE*) *partition*). The triangle is bisected by its longest edge, followed by the edge bisection of the resulting triangles by the remaining original edges of t.

The angles of the triangles generated by this partition are uniformly bounded away from 0 and π. Additional refinement of adjacent triangles is necessary to ensure the conformity of the mesh, but this refinement is also made based on bisecting the longest edge.

Definition 3 (*Newest vertex bisection*). The newest vertex bisection of a triangle t is obtained by connecting one of the vertices, called *peak*, to the midpoint of the opposite edge, called the *base*.

The peak is usually chosen to be the opposite vertex to the longest edge. Note that once the peak (or the base which is always the opposite edge to the peak) has been selected in the initial triangle, the partition works in all the successors without any computation. It is easily shown that only four similarity classes of triangles and only eight distinct angles are created by this method, so the angles satisfy the important condition of being bounded away from 0 and π.

Note that implementing the LE and 4T-LE partitions require the computation of the lengths. Also, very often the 4T-LE partition of a triangle and its successors is equivalent to two successive steps of the Mitchell partition. It happens when the bisected edge in the newest vertex bisection, coincides with the longest-edge [9].

In three dimensions several techniques have been developed in the last years for refining (and coarsening) tetrahedral meshes. Algorithms based on pure longest-edge bisection of tetrahedra have been developed by Rivara and Levin [17], Muthukrishnan et al. [12], etc.

Definition 4 (*Single bisection and LE bisection*). Single bisection consists of dividing the tetrahedron into two sub-tetrahedra by the midpoint of one of the edges. When the longest edge is chosen to bisect the tetrahedron, we say that the longest edge (LE) bisection has been done.

Other tetrahedra partitions which are not longest-edge based partitions have been developed in the last years. They divide the edges into two edges, the triangular faces into four triangles, and the tetrahedra into eight son-tetrahedra. Lastly, a partition in eight tetrahedra based on the length of the edges, the 8-tetrahedra longest-edge partition, has been investigated and used for refining and coarsening tetrahedral meshes [14,15].

Definition 5 (8*-tetrahedra longest-edge* (8*T-LE*) *partition*). For any tetrahedron t, the *8T-LE* of t produces 8 sub-tetrahedra by performing the 4T-LE partition of the faces of t, and by subdividing the interior of the tetrahedron t consistently with the division of the faces.

The 8T-LE partition can be achieved by performing a sequence of bisections by the midpoints of the edges of the original tetrahedron taking into account the length of the edges as follows:

Theorem 6. *For any tetrahedron t of unique longest-edge, the* 8*T-LE partition of t is obtained as follows*:

(1) *Longest edge bisection of t producing tetrahedra t_1, t_2.*

(2) *Bisection of t_i, for $i = 1, 2$, by the longest edge of the common face of t_i with the original tetrahedron t, producing tetrahedra t_{ij}, for $j = 1, 2$.*
(3) *Bisection of each t_{ij} by the midpoint of the unique edge equal to an edge of the original tetrahedron.*

Fig. 1 shows the four refinement patterns given by the relative positions of the longest edges of the faces of the tetrahedron. Note that the 8T-LE partition is the extension to 3 dimensions of the 4T-LE partition, in the sense that this partition takes into account the longest-edge of the tetrahedra and their faces.

One of the approaches based on bisection is that of Liu and Joe [6,7]. This partition can be understood as the 3D version of the Mitchell partition. The edges for bisection are chosen without any computation following a rule between the edge types involved and their relative position [6,7] to automatically assign the types to the new edges. The bisection edge is chosen without any computation and following a rule between the type edges involved and their relative position to automatically assign the types to the new edges.

Definition 7 (*Liu–Joe partition*). Their partition can also be explained by means of a mapping between the original tetrahedron T and a canonical tetrahedron P with the same volume as T [7]: Let T be any tetrahedron and let P be the special tetrahedron with the same volume as T. Then

(a) Transform T to P by the affine transformation $y = M^{-1}(P, T)x + b_0$.
(b) Iteratively bisect P to three levels, always bisecting the longest-edge.
(c) Transform the 8 sub-tetrahedra P_i of P back to sub-tetrahedra T_i of T using the inverse affine transformation $y = M(P, T)x + b_1$.

Fig. 2 shows the partition in eight tetrahedra of the canonical Liu–Joe tetrahedron. This tetrahedron has the interesting property that its partition in 8 sub-tetrahedra yields in 8 similar tetrahedra to the former one.

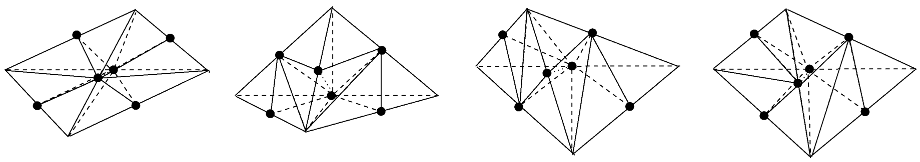

Fig. 1. Refinement patterns for the 8T-LE partition.

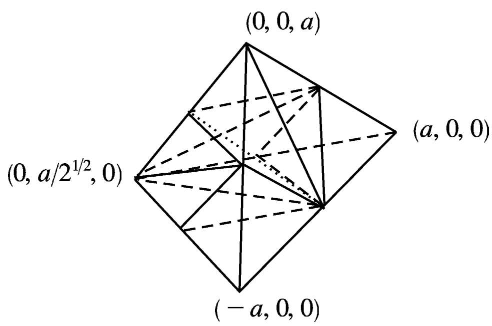

Fig. 2. Canonical Liu–Joe tetrahedron.

Since in the sub-tetrahedra of T the longest-edge may not be the one that is bisected, this partition is not equivalent to three levels of the longest-edge partition. On the other hand, since the 8T-LE partition is based in the application of the 4T-LE partition on the faces of the tetrahedron and it takes into account the length of th edges, both partitions are not equivalent in general.

3. Comparison and discussion of Rivara–Levin, Liu–Joe and Plaza–Carey algorithms

Liu–Joe and Plaza–Carey classify the tetrahedra depending on the relative positions of the edges.

Let us consider the generic tetrahedron with nodes and edges numbered as in Fig. 3 so that edge $(3, 4)$ is the longest one. The faces are numbered with the number of the opposite vertex, as usual. Introducing a criterion similar to that of Liu and Joe [7], Bänsch [1] or Mukherjee [11], we assign to each edge of the tetrahedron a *type*-label between 1 and 3: the longest edge of a tetrahedron is labeled type 1. Note that this edge is also the longest edge of two faces (faces 1 and 2 sharing the edge). The longest edge of each one of the other two faces (faces 3 and 4) is labeled edge type 2, and the rest of the edges are type 3. Edge type 1 is the reference edge of the tetrahedron.

The classification of a tetrahedron following Plaza–Carey algorithm is showed next [14]:

Procedure *Classification*
/* Input variables: t tetrahedron
  Output variables: type */
  If edge 6 has type 3 **then**
    t has type 1
  Else If edge 6 $=$ *longest*(face 3) $=$ *longest*(face 4) **then**
    t has type 2
  Else
    t has type 3
  End If.

The classification of a tetrahedron following Liu–Joe algorithm is as follows: For each triangular face of a tetrahedron a marked point is defined to be the midpoint of its longest edge. If two faces of a tetrahedron share the same *marked point*, the marked point is called *doubly marked point* of the tetrahedron; otherwise, it is called *singly marked point*. It is obvious that the midpoint of the longest edge of a tetrahedron must be a doubly marked point. Considering the relative position of the edges of

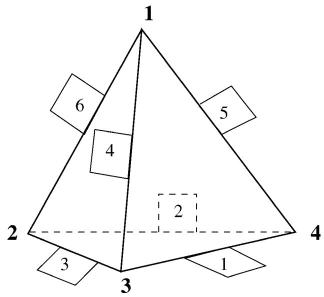

Fig. 3. Tetrahedron in canonical position and local numeration of the vertices and edges.

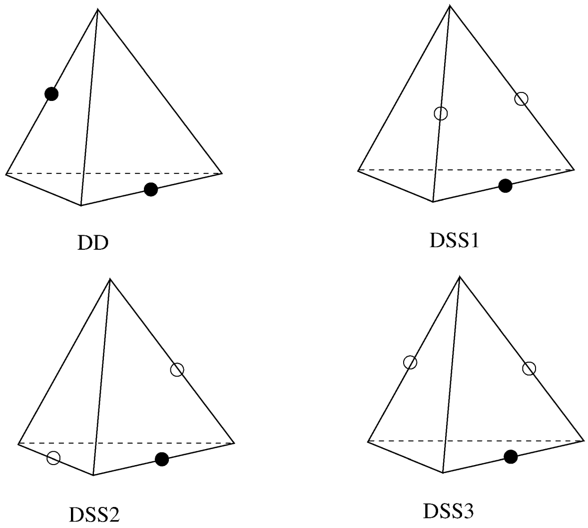

Fig. 4. Liu–Joe classification of the tetrahedra.

a tetrahedron and the number of marked points, Liu–Joe distinguish four kind of tetrahedra called DD, DSS1, DSS2 and DSS3, see Fig. 4. Looking at the first three levels of bisection of a tetrahedron T, the edge with doubly marked point is bisected in the first bisection; then in each sub-tetrahedra of T the bisected edge is chosen as the with a marked point among the edges of T; in the third bisection the three remaining edges of T are bisected.

For the successive levels of bisection, the classification of the edges is automatically inherited from the classification of the edges in the first step of bisection. In the Liu–Joe algorithm, the number of similarity classes is proved to be bounded and therefore the meshes cannot degenerate. However, Plaza–Carey algorithm has not yet a count and proof of the number of similarity classes and only a numerical support exists for this issue.

DD: Tetrahedron t has two doubly marked points on a pair of opposite edges.

DSS1: Tetrahedron t has one doubly marked points and two singly marked points; these points are in the same face.

DSS2: Tetrahedron t has one doubly marked points and two singly marked points; these points are on a pair of opposite edges.

DSS3: Tetrahedron t has one doubly marked points and two singly marked points; one of them is on an edge opposite to the edge with doubly marked point.

Both classifications are equivalent as it is showed in Table 1.

3.1. Comparison of degeneracy

A tetrahedron shape measure is a continuous function that evaluates the quality of a tetrahedron. It must be invariant under translation, rotation, reflection and uniform scaling of the tetrahedron, maximum for the regular tetrahedron and minimum for a degenerate tetrahedron. There should be no local maximum

Table 1
Classification of tetrahedra

Plaza and Carey	Liu and Joe
type 2	DD
type 1	DSS1
type 1	DSS2
type 3	DSS3

other than the global maximum for a regular tetrahedron and there should be no local minimum other than the global minimum for a degenerate tetrahedron [4].

As it is well known, there are several measures of mesh quality (or degeneracy) but in this case we are interested in two of them, the angle Φ_P introduced by Rivara and Levin [17] and the measure η introduced by Liu and Joe [6,7]. For tetrahedron t and each vertex $P \in t$, Rivara and Levin use the measure Φ_P associated with the solid angle at P:

$$\Phi_P = \sin^{-1}\bigl(1 - \cos^2\alpha_P - \cos^2\beta_P - \cos^2\gamma_P + 2\cos\alpha_P\cos\beta_P\cos\gamma_P\bigr)^{1/2}, \tag{1}$$

where α_P, β_P, γ_P are the associated corner angles. The relation between Φ_P and the solid angle Ω_P at P, can be established as follows. From [6] we get:

$$\sin(\Omega_P/2) = \frac{(1 - \cos^2\alpha_P - \cos^2\beta_P - \cos^2\gamma_P + 2\cos\alpha_P\cos\beta_P\cos\gamma_P)^{1/2}}{4\cos(\alpha_P/2)\cos(\beta_P/2)\cos(\gamma_P/2)}. \tag{2}$$

So,

$$\Omega_P = 2\sin^{-1}\frac{\sin(\Phi_P)}{4\cos(\alpha_P/2)\cos(\beta_P/2)\cos(\gamma_P/2)}. \tag{3}$$

Liu and Joe introduce the estimate

$$\eta(t) = \frac{12(3\ \text{volume})^{2/3}}{\sum_{i=1}^{6} l_i^2}, \tag{4}$$

where l_i means the length of the edge i.

We refer to the reader to reference [14] for an empirical study and comparison on the behavior of $\eta_{\min}$, the minimum value of η in the tetrahedra of the mesh, and $\Omega_{\min}$, when the pure longest-edge bisection based refinement, the Liu–Joe partition, and the 8T-LE partition are applied to different shaped tetrahedra. It is worth to be noted here that the global application of the 8T-LE and Liu–Joe partitions to any initial conforming mesh yield a conforming mesh. However, the longest-edge bisection needs additional refinements in order to get a conforming mesh.

The results using 8T-LE partition were comparable to those obtained by the longest edge bisection of Rivara and Levin [17] and by the Liu and Joe partition [7].

3.2. Comparison of coefficients of non-degeneracy

Our goal is to obtain an experimental coefficient of non-degeneracy for the 8-tetrahedra longest-edge partition and to compare it with the theoretical value obtained by Liu and Joe [6]. Although we do not provide examples with a complex initial mesh taken from a real application, the wide variety of shapes

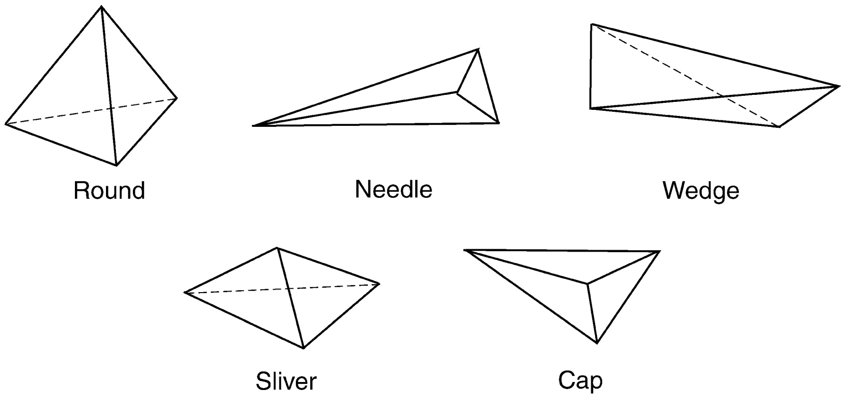

Fig. 5. Shape-based types of tetrahedra.

in the studied tetrahedra allows us to conjecture that the non-degeneracy coefficient will be very close to the empirical non-degeneracy coefficient given in this paper.

For this purpose we are interested in studying some common shape-based types of tetrahedra that may occur in refined meshes [2,3,13]. We give descriptive names to the five different types of three-dimensional simplices: A *round* tetrahedron has no bad angles of any kind. A *needle* or *thin* has one small solid angle. A *wedge-like* element has small but not large dihedrals and no large angles of any kind. A *sliver* has small and large dihedrals, but no large solid angle. A *cap-like* tetrahedron has a large—nearly flat—solid angle. For the cap tetrahedron, the circumscribed sphere's radius is hence much larger than the longest edge. Fig. 5 shows some common shape-based types of tetrahedra.

Rivara and Levin in [17] reported an empirical study of the reduction of the solid angle size due to repeated subdivision by the pure longest-edge bisection of tetrahedra. Their results allowed to conjecture that the shape function associated to the minimum angle $\Phi_{\mathrm{MIN}}(k)$, where k is the bisection iteration, converges asymptotically to a fixed value.

Let t be an initial tetrahedron in which the 8T-LE partition is applied. Thus, from the initial mesh $\tau^1 = \{t = t_1^1\}$, mesh $\tau^2 = \{t_i^2\}$ is obtained. The successive application of the 8T-LE partition to any tetrahedron and its successors yields an infinite sequence of tetrahedral meshes $\tau^1, \tau^2, \tau^3, \ldots$.

We give here experimental evidence showing that for a standard shape measure etha (η), the non-degeneracy convergence to a fixed positive value is guaranteed, that is, for any tetrahedron t_i^n in τ^n, $n \geqslant 1$,

$$\eta\left(t_i^n\right) \geqslant c\eta(t), \tag{5}$$

where c is constant and independent of i and n. With this aim, we perform seven 8T-LE global refinements beginning with an initial mesh $\tau^1 = \{t_1^1\}$. This initial mesh contains one different initial tetrahedron in each case.

The refinement process ends for the 8T-LE partition with the mesh level τ^8 containing $8^7 = 2,097,152$ elements, while for the Rivara–Levin partition the process ends with mesh level τ^{15}. As a quality measure of τ^n we define $\eta(\tau^n) = \min\{\eta(t_i^n)\}$ for all $t_i^n \in \tau^n$. Then the value

$$c_n = \frac{\eta(\tau^n)}{\eta(\tau^1)} \tag{6}$$

is calculated. This value is taken as an estimation of c in Eq. (5).

First experiment. This experiment considers a Cap tetrahedron as initial mesh with coordinates $(0,0,0)$, $(2\sqrt{3},0,0)$, $(\sqrt{3},3,0)$ and $(\sqrt{3},1,\frac{\sqrt{2}}{2})$. We perform seven 8T-LE global refinements. The finest mesh τ^8 contains $8^7 = 2{,}097{,}152$ elements.

Second experiment. This experiment considers an initial tetrahedron called Wedge tetrahedron with coordinates $(0,0,0)$, $(2\sqrt{3},0,0)$, $(\sqrt{3},3,0)$ and $(\frac{\sqrt{3}}{20},\frac{1}{20},\frac{\sqrt{2}}{10})$ as initial mesh, and proceeds as the first experiment. Note that the results are analogous to those in experiment 1.

Third experiment. This experiment is analogous to those above, but in this case we have taken as initial mesh a tetrahedron called Sliver with coordinates $(0,0,0)$, $(2\sqrt{3},0,0)$, $(\sqrt{3},3,0)$ and $(\sqrt{3},-1.819708028,0.2219158345)$.

Fourth experiment. In this case a Needle tetrahedron is taken as initial mesh, with coordinates $(0,0,0)$, $(2\sqrt{3},0,0)$, $(\sqrt{3},3,0)$ and $(\sqrt{3},1,40\sqrt{2})$, and the same procedure is repeated.

Table 2 shows different results we have obtained.

It should be noted that the most unfavorable case in all the experiments corresponds to the regular tetrahedron as initial mesh. This fact is in accordance with the behavior of the 4T-LE partition in two dimensions, in which the lowest bound of degeneracy for the angles is reached ($\alpha_n = \frac{1}{2}\alpha_0$) for the equilateral triangle. On the other hand, in the last two experiments, the best case for the degeneracy constant corresponds to the worst initial tetrahedron, sliver tetrahedron and needle tetrahedron.

At this stage, we are interested to repeat the experiments above, but using now the Rivara–Levin partition and to get an experimental estimation of c. The finest mesh τ^{15} contains different number of elements but with same order of the experiments above. Results are summarized in Table 3.

Table 2
Shape measures (η), mesh τ^8 with 2, 097, 152 elements for 5 different initial tetrahedra t_1^1 using the 8T-LE partition

t_1^1	$\eta(t_1^1)$	$\min\{\eta(t_i^8)\}$	$c_8 = \frac{\eta(\tau^8)}{\eta(\tau^1)}$	tetrahedra
Regular	1.000	0.31553	0.31553	2 097 152
Cap	0.203	0.11622	0.57158	2 097 152
Wedge	0.166	0.09191	0.55390	2 097 152
Sliver	0.183	0.14484	0.79030	2 097 152
Needle	0.055	0.04067	0.73958	2 097 152

Table 3
Shape measures (η), mesh τ^8 5 different initial tetrahedra t_1^1 using the Rivara–Levin partition

t_1^1	$\eta(t_1^1)$	$\min\{\eta(t_i^{15})\}$	$c_8 = \frac{\eta(\tau^{15})}{\eta(\tau^1)}$	tetrahedra
Regular	1.000	0.21413	0.21413	1 717 072
Cap	0.203	0.00239	0.01783	2 031 436
Wedge	0.166	0.07823	0.47145	1 952 996
Sliver	0.183	0,04121	0.22485	2 863 125
Needle	0.055	0.03277	0.59611	2 119 488

Table 3 shows that the most unfavorable case in all the experiments corresponds to the cap tetrahedron as initial mesh and the best values are for the wedge and needle tetrahedra.

We deduce that the estimated value for the non-degeneracy constant is $c = 0.31553$ for the 8T-LE partition. However, Liu–Joe and Rivara–Levin partition get the value $c = 0.1417$ and $c = 0.01783$ respectively.

4. Conclusions

In this paper three of the main refinement algorithms based on bisection in 3D have been reviewed and compared.

(1) In some particular cases Rivara–Levin, Liu–Joe and Plaza–Carey partitions are equivalent.
(2) Liu–Joe partition and Plaza–Carey partition are stable in the number of tetrahedra, but Rivara–Levin partition is not.
(3) Liu–Joe have proved the non-degeneracy of their partition with non-degeneracy constant $c = 0.1417$.
(4) Empirical studies for the non-degeneracy constant for the Plaza–Carey partition improves the Liu–Joe results. The minimum value obtained $c = 0.31553$, corresponds to the regular tetrahedron as initial mesh. On the other hand, for the Rivara–Levin partition the minimum value we have obtained is $c = 0.01783$, for a cap tetrahedron as initial mesh.

Acknowledgements

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