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Computer Aided Geometric Design 23 (2006) 684–697
COMPUTER AIDED GEOMETRIC DESIGN
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Block-balanced meshes in iterative uniform refinement

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Received 25 February 2006; accepted 16 September 2006

Available online 1 November 2006

Abstract

In this paper we study and delimit a property of known four triangle subdivisions that is useful in tri to quad mesh conversion methods. We provide both theoretical results and empirical evidence showing that iterative application of the four triangles longest-edge subdivision and the four triangles similar subdivision produces block-balanced meshes, meshes in which triangle pairs sharing a common longest edge tend to cover the area of the entire mesh. Some other properties of such triangle subdivisions regarding mesh quality and adjacency relationships are also discussed.

Keywords: Mesh refinement; Subdivision; Tri to quad

1. Introduction

Engineers and designers need to feel confident in the results of their analyses before sending a product to prototyping or production. The meshes need to be refined based upon error norms and other information output by the solver. Mesh refinement is also a key tool in adaptive tessellation of NURBS surfaces (Kumar, 2000). In this sense, Delaunay meshes have been widely used since they avoid long and skinny triangles and produce the maximum possible smallest-internal angle of any triangle (Bern and Eppstein, 1995; Shewchuk, 2002b). Refinement techniques are also used for enhancement of meshes obtained from trimmed NURBS surfaces, see an application in (Rabi Kumar et al., 2001). The number of triangles can be further increased/decreased depending on the application requirement.

In engineering applications, triangular and quadrilateral meshes are commonly used for Finite Element Analysis. Note that the terminology 'quadrangle' and 'quadrangulation' is equivalent to 'quadrilateral' and 'quadrilateralization' (Ramaswami et al., 1998). Quadrangle and quadrilateral are used interchangeably in the literature as well as 'triangle' and 'triangulation'. For brevity it is often used tri and quad.

In Computer Graphics, Virtual Reality and CAD applications, triangular and quadrilateral meshes have emerged as a common and most versatile free-form surface representation. For realistic objects it is often necessary that these meshes consist of a large number of faces. For this reason, are usually used refinement techniques to add interpolated

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0167-8396/$ – see front matter
doi:10.1016/j.cagd.2006.09.003

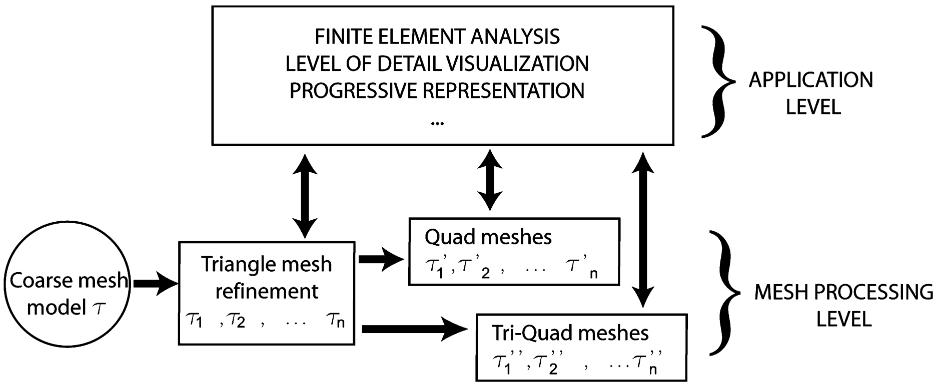

Fig. 1. Mesh Processing level and interaction with the Application Level.

points to the mesh. For example, hierarchical representations can be obtained via mesh refinement and this can be used to generate Level of Details (LOD) or Progressive Representations (Hoppe, 1997; Luebke, 2001).

1.1. Paper contribution

Our paper introduces the idea of block-balanced meshes and focus on how iterative triangle refinement not only produces meshes with an increasing number of basic blocks but also presents shape improvement properties for the new triangles generated. Its main contribution is a theoretical analysis of the behavior of two common refinement schemes when iterative refinement is applied to a base triangle mesh. We provide both theoretical results and empirical evidence showing that successive application of the four triangles longest-edge subdivision and the four triangles similar subdivision to an arbitrary unstructured triangular mesh produces meshes in which the triangle blocks sharing a common longest edge tend to cover the area of the entire mesh. This property is of relevance to develop efficient tri to quad mesh conversion algorithms by merging basic blocks (Ramaswami et al., 1998; Velho, 2000; Velho and Zorin, 2001). Some other properties regarding mesh quality and adjacency relationships are also discussed for these triangle subdivision methods.

To illustrate some scenarios in which our work is of interest we present in Fig. 1 the Mesh Processing level as it interacts with the Application level of computer applications in the field of Finite Element Analysis (FEA), Level Of Detail (LOD) visualization and Progressive Representation (PR) (Hoppe, 1997; Luebke, 2001). The Application level relies on discretization methods that for example, starting from a coarse mesh, refine the mesh using triangle subdivision methods several times to obtain a sequence of new meshes with a greater level of triangles. This sequence of meshes are then used to provide the Application level with a top-down list of mesh models with distinct geometric/topological features that are adapted to the specific requirements at the Application level. For example, an end-user may want to solve a PDE problem using a sequence of nested meshes accompanied of a Multigrid solver (Hackbush, 1985). With the aim of solution analysis, convergence etc, same solving process may be carried out on a different underlying mesh, either a quad-based or tri-quad based mesh and so, tri to quad conversion methods are desirable tasks within the Mesh Processing level.

The paper is organized as follows. Section 1.2 gives a short background of triangle to quadrangle conversion strategies. Section 2 presents some preliminaries in triangle refinement and introduces the four triangle subdivision methods used in the paper. Section 3 studies adjacency and valences properties of such triangle subdivisions. In Section 4 we introduce the Block-balanced meshes and prove our main result. We follow in Section 5 with an application of our main result to convert tri to quad meshes. We conclude in Section 6.

1.2. Tri to quad mesh conversion

The indirect approach to quadrilateral mesh generators generates quadrilaterals by merging triangles in the background triangle mesh. More precisely, the idea behind obtaining a quadrangulation in this fashion is to pair up neighboring triangles to form quadrangles (Ramaswami et al., 1998; Velho, 2000). The indirect approach then benefits from the simplicity and from the possibility to handle simultaneously either a tri or a quad mesh. In addition, the quality of the triangulation influences the quality of the quadrilateral mesh. Following this approach, a full tri to quad is not always possible and so a number of triangles could not be paired up. Hence, it is necessary the introduction of additional points to the mesh. The goal of converting a triangular a mesh to a quadrilateral one is to find the maximum possible number of such blocks. This is also known as the problem of finding the maximum cardinality matching in the dual graph of a triangulation. In this sense Ramaswami et al. (1998) stated that a triangulation admits a quadrangulation without additional points if and only if the dual graph of the triangulation admits a perfect matching.

Rather than find the maximum possible number of triangle blocks, our approach here is to exploit a property of iterative mesh refinement. Proofs are included here to show that iterative refinement naturally generates basic blocks and that the refined meshes converge to a triangle mesh with a maximal matching property. It should be noted that mesh refinement is mainly satisfactory for applications that need a hierarchy of nested refined meshes, as Finite Element Methods, Level of Details, Progressive Representations, etc. see Fig. 1. However, for the case of a tri to quad mesh conversion in which no refinement is needed, since iterative refinement introduces a large number of vertices and triangles, better results are obtained by the application of well-known direct methods for quad conversion such those proposed in (Ramaswami et al., 1998).

2. Preliminaries in triangle refinement

In CAGD, subdivision surfaces started as a generalization of uniform splines (de Boor, 1978). The key idea of a subdivision scheme draws upon knot insertion techniques (Lane and Riesenfeld, 1980), and has its roots in the 'de Boor' algorithm (de Boor, 1978). Nonetheless, the beginning of the field is identified with the development of the first subdivision surfaces for irregular meshes. Catmull and Clark (1978) and Doo and Sabin (1978) extended bicubic and biquadratic B-splines, respectively, to arbitrary meshes that generalize quadrilateral meshes.

A problem of much interest is the refinement of meshes. Iterative refinement can be seen as the iterative application of any subdivision scheme to a given mesh. Refinement produces a new triangle nested mesh suitable, for example, for a new calculation step in the Finite Element Method. If τ_0 is an initial triangular mesh of a bounded domain Ω then the iterative refinement on τ_0 produces a sequence of nested meshes $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$. The hierarchical sequence of meshes with the underlying nested sequence of basis functions permits adaptive solution until a given error is achieved in the context of Finite Element computations.

Many subdivision schemes and associated refinement algorithms for triangular meshes have been proposed and studied (Carey, 1997; Ivrissimtzis et al., 2004). The simplest subdivision scheme is *Bisection* into two subtriangles by connecting the midpoint of one of the edges to the opposite vertex, Fig. 2(b). If the Longest Edge (LE) is chosen for the bisection, then this is called *Longest Edge Bisection*. Longest-edge based algorithms have been used solely (Rosenberg and Stenger, 1975) or combined with Delaunay triangulation for the quality triangulation problem (Rivara and Iribarren, 1996; Shewchuk, 2002a).

The *Four Triangles Longest Edge Subdivision* (*4T-LE*) bisects a triangle into four subtriangles where the original triangle is first subdivided by its longest edge as before and then, the two resulting triangles are bisected by joining the new midpoint of the longest edge to the midpoints of the remaining two edges of the original triangle, see Fig. 2(c). For Longest-Edge and 4T-LE iterative refinement, a lower bound of the smallest angle, $\alpha_{\min}$, (min-min condition) generated is: $\frac{\alpha_0}{2} \leqslant \alpha_{\min}$ (Rivara and Iribarren, 1996; Rosenberg and Stenger, 1975), where α_0 is the minimum interior angle in the initial triangle, and this is relevant for obtaining non-degenerate triangle meshes. Moreover, in (Rivara and Iribarren, 1996) it has been proved that in the iterative 4T-LE refinement of a mesh of obtuse triangles, the smallest angles monotonically increase, while the largest angles decrease in amount (at least) equal to the smallest angle of the last refined mesh. The iterative 4T-LE subdivision then produces a finite sequence of 'better' triangles satisfying good properties regarding mesh quality (Plaza et al., 2004; Rivara and Iribarren, 1996). The subdivision scheme for the 4T-LE subdivision is quite similar to the 4-8 refinement of (Velho, 2001; Velho and Zorin, 2001). However, rather than apply two separate LE bisections, our scheme simply gets the longest edge of a triangle and then a midpoint reconnection is made.

The *4-Triangles Similar subdivision* (*4T-S*) of a triangle t_0 is obtained by joining the midpoints of the edges of t_0 with segments parallel to the edges, see Fig. 2(d). The advantage of this subdivision scheme compared to 4T-LE is that no previous edge-length computation needs to be made. Furthermore, it is clear that the application of the 4-Triangles Similar subdivision to any initial triangle yields four similar triangles to the former one. Hence, the iterative application of the 4-Triangles Similar subdivision to any initial triangle does not generate distinct (up to

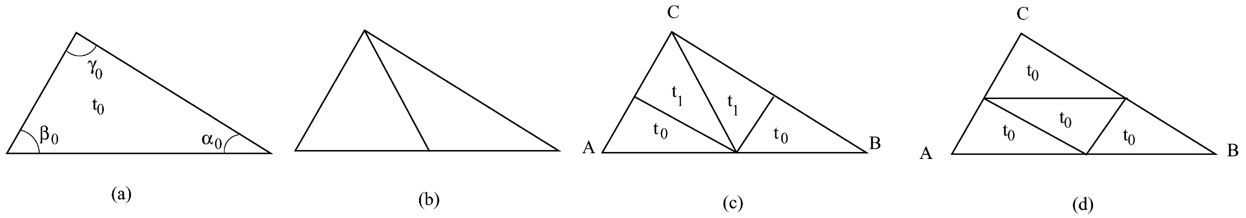

Fig. 2. (a) Initial triangle t_0, (b) LE subdivision of triangle t_0, (c) 4T-LE subdivision of triangle t_0 and (d) 4T Similar subdivision of t_0.

similarity) triangles, which is a very good feature for the application of this subdivision to initially well shaped meshes.

As we have mentioned above 4T-LE and 4T-S pose very nice properties and so we shall focus on these subdivisions throughout this work. Next section presents some useful adjacency properties of such subdivisions.

3. Adjacency properties of triangular meshes through iterative uniform refinement

Let τ_0 be any initial conforming triangulation with N_0 vertices, E_0 edges and T_0 triangles. A mesh is conforming if the intersection of adjacent triangles is either a common vertex or an entire side. Then, after n applications of either the 4T-LE subdivision or the 4T-S subdivision to each triangle of τ_0 and its descendants, producing a globally refined and conforming triangulation τ_n, the number of vertices, edges and triangles in τ_n (respectively, N_n, E_n and T_n) are related with the number of elements in the preceding triangulation τ_{n-1} by means of the following constitutive equations:

$$\begin{aligned} N_n &= N_{n-1} + E_{n-1} \\ E_n &= 2E_{n-1} + 3T_{n-1} \\ T_n &= 4T_{n-1} \end{aligned} \tag{1}$$

Similar constitutive equations also hold in a more general class of subdivisions, called skeleton-regular simplex partitions, (Plaza and Rivara, 2002). Other example in this group is the barycentric subdivision, where the triangle refinement replaces every triangle by three, by joining every vertex to the barycenter of the triangle.

Proposition 1. (Plaza and Rivara, 2002) *For any 2D conforming triangulation having N_n vertices, E_n edges and T_n triangles, the average number of triangles by vertex (vertex valence) and edges by vertex are given as follows*:

$$Av\#(triangles\ per\ vertex) = \frac{3T_n}{N_n}$$

$$Av\#(edges\ per\ vertex) = \frac{2E_n}{N_n}$$

Proposition 2. (Plaza and Rivara, 2002) *Let τ_0 be a (conforming) triangular mesh. For the 4T-LE and 4T-S subdivision let N_n, E_n, and T_n be the total number of vertices, edges, and triangles, respectively, after the nth subdivision application. Then the asymptotic average of non-trivial adjacency numbers (noted as As Av# for Asymptotic Average Number of) of topological elements are independent of the particular subdivision of each triangle and these numbers are as follows*:

$$As\ Av\#(triangles\ per\ vertex) = \lim_{n\to\infty} \frac{3T_n}{N_n} = 6$$

$$As\ Av\#(edges\ per\ vertex) = \lim_{n\to\infty} \frac{2E_n}{N_n} = 6$$

In order to calculate the asymptotic average adjacencies of the topological components of the 4T-LE and 4T-S subdivision and so prove Proposition 2, we need to solve its associated constitutive equations (Plaza and Rivara, 2002). This can also be done either by generation functions or by using a symbolic calculus package like *Maple* or *Mathematica*.

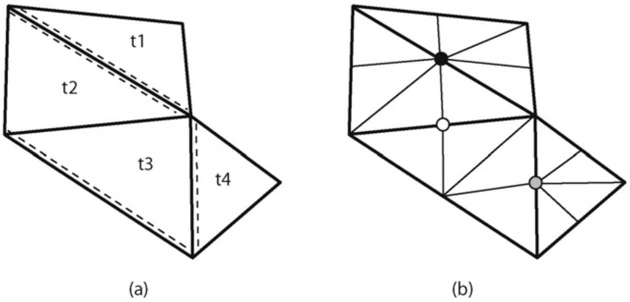

Fig. 3. Refinement of a simple mesh by the 4T-LE subdivision, and valences for the interior new nodes.

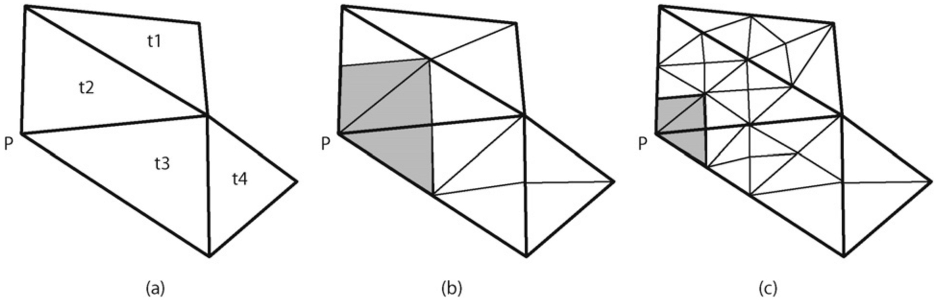

Fig. 4. 4T-LE refinement around vertex P and stable molecule in shaded color.

However, from a practical point of view it should be noted that the 4T-S subdivision is ideal, because the number of non-6 vertices remains constant, as an ever decreasing fraction of the total number. The new internal vertices (i.e. non-boundary) of the 4T-LE subdivision are of valence 4, 6 or 8 depending on whether or not that vertex belongs to a common longest edge for two adjacent triangles. In Fig. 3 a simple mesh with four triangles is refined by the 4T-LE subdivision. Internal nodes present valences 4 (white node), 6 (grey node) and 8 (black node). Note that for an old vertex P, the valence changes by the application of 4T-LE subdivision if at least one of the triangles sharing vertex P has its greater angle at P, since 4T-LE subdivision always divides the greater angle of a triangle, see Fig. 2(c).

To study the maximum valence in a 4T-LE refinement the concept of *stable molecule* is given below:

Definition 1. For any conforming triangulation τ and any vertex P of τ, the stable molecule associated with vertex P is the partition of the plane around vertex P, induced by the 4T-LE refinement of each triangle around P, such that further 4T-LE refinements of the triangles around P do not change the number of triangles sharing P.

Fig. 4 illustrates the concept of stable molecule associated to vertex P. Note that the 4T-LE subdivision of a triangle never divides the two smallest angles of each triangle, and the number of angles converging in P are the same once the stable molecule of P has been achieved (Fig. 4(b)). After that, further refinement around P does not change the shape of the triangles sharing P but only their size, with $1/2$ scaling factor. Rivara and Inostroza (1997) established the following result in the context of longest-side refinement, also valid for the 4T-LE refinement:

Proposition 3. *Let τ any conforming triangulation and consider any vertex P of τ. After a finite number of iterations of the longest-side refinement of the triangles around P, the stable molecule associated with vertex P is obtained. After that, the next iterations of the refinement do not partition the angles of the stable molecule.*

Note that the maximum valence for an internal node P is equal to the number of triangles belonging to its stable molecule. This number depends on the shape of the triangles sharing P and it is related with the number of dissimilar triangles generated by the 4T-LE subdivision, see references (Plaza et al., 2004, 2005).

Consider now, as another example, meshes in which only pairs of right triangles sharing their common longest edge are presented, see Fig. 5(a). Note that all the internal vertices have valence 6. Since every 1-neighborhood of

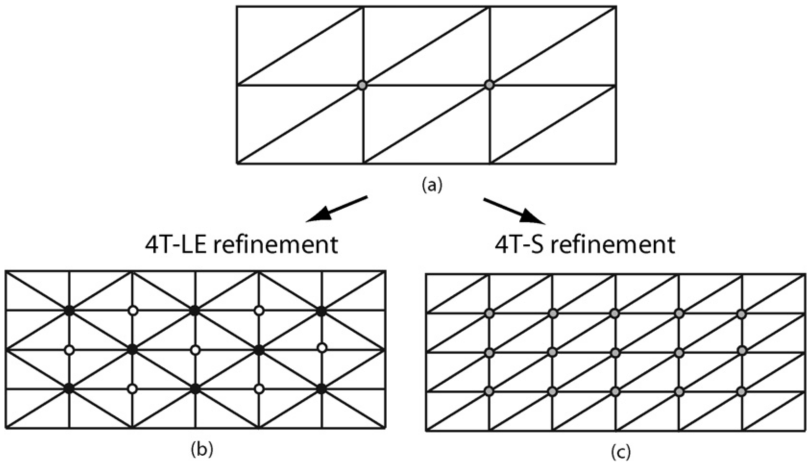

Fig. 5. (a) Regular right 6 mesh. (b) 4T-LE refinement $[4.8^2]$ Laves tiling. (c) 4T-S refinement.

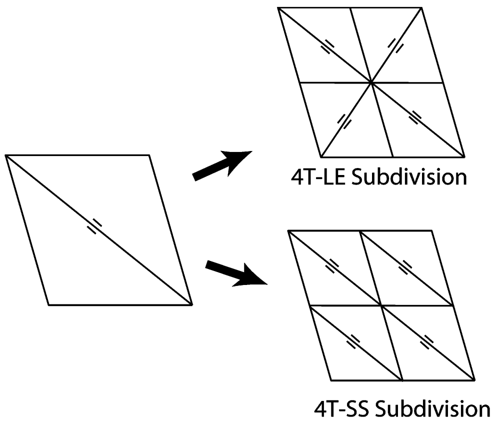

Fig. 6. Basic blocks generation in 4T-LE and 4T-S subdivision.

every internal vertex (of valence 6) has only neighbors of valence 6, these regular 6 meshes correspond to a $[6^3]$ Laves tiling (Grünbaum and Shephard, 1987; Velho, 2001). Note that the 4T-LE uniform refinement of such a mesh yields a 4-8 mesh, that is, a mesh in which only vertices of valence 4 and 8 are presented, and moreover where the 1-neighborhood of every internal vertex of valence 4 has only neighbors of valence 8, and the 1-neighborhood of every internal vertex of valence 8 consists of a ring of vertices with alternating valences 4 and 8. These regular 4-8 meshes correspond to a $[4.8^2]$ Laves tiling (Grünbaum and Shephard, 1987; Velho, 2001), see Fig. 5(b). Finally, observe that the 4T-S uniform refinement of a regular right mesh yields another regular right mesh, see Fig. 5(c). In this paper, in some sense, these kind of meshes are generalized by the introduction of the concept of *basic block*.

4. Block-balanced meshes in the 4T-LE and 4T-Similar refinement

Some definitions are given in order: Two neighbor triangles (t, t^*) will be a *basic block* if the common edge is the longest edge of (t, t^*). In (Velho and Zorin, 2001) a similar definition is given for basic blocks in which the common edge is not necessarily the longest edge. If a triangle t does not belong to a basic block, t is said to be a 'single' triangle. A triangulation τ is said to be Block-balanced if it is comprised of basic blocks.

Let τ contain T triangles with B basic blocks. Then, the Block-balancing degree of τ, noted as $D(\tau)$, is defined as $D(\tau) = 2B/T$. Note that, since $0 \leqslant 2B \leqslant T$, then $0 \leqslant D(\tau) \leqslant 1$. Fig. 6 shows the generation of basic blocks through the 4T-LE and 4T-S subdivision schemes respectively.

Remark 1. The 4T-LE subdivision (and also the 4T-S) of two triangles in a block produce eight triangles in four blocks.

In Fig. 5(a), a Block-balanced mesh is shown. In this mesh, if one applies either the 4T-LE or the 4T-Similar uniform refinement, then all triangles are still basic blocks. Moreover, a full quad conversion can be obtained simply removing interior edges of basic blocks.

Our next goal is to prove that the uniform application of either the 4T-LE or the 4T-Similar subdivision will produce a sequence of meshes with increasing Block-balancing degree approaching 1.

4.1. The 4T-LE subdivision case

Proposition 4. *If the 4T-LE subdivision of an initial triangle t_0 introduces two triangles t_1 in a basic block, then iterative application of the 4T-LE subdivision therefore introduces basic blocks excepting the triangles located at the longest edge of t_0. Moreover, in this case only two classes of similar triangles are generated, corresponding to t_0 and t_1 respectively* (*see Fig.* 2(c)).

Proof. The hypothesis of Proposition 4 is depicted in Fig. 2(c). The proof follows trivially from the angle properties of parallel lines in the nested triangles. □

To demonstrate that recursive uniform 4T-LE refinement introduces meshes with relatively more basic blocks for any arbitrary triangular mesh we consider right, acute and obtuse triangles respectively. We begin in the next proposition with the right and acute triangle cases:

Proposition 5 *(Right and acute triangle cases). The application of the 4T-LE subdivision to an initial right or acute triangle t_0 produces two new single triangles similar to the original one* (*located at the longest edge of t_0*) *and a basic block of triangles t_1. These triangles t_1 are also similar to the original one t_0 in the case of right triangle t_0, and they are similar to each other but non-similar to the initial one in the case of acute triangle t_0.* (*See Fig.* 2(c).)

The obtuse triangle case offers a different situation:

Proposition 6 *(Obtuse triangle case). The application of the 4T-LE subdivision to an initial obtuse triangle t_0, produces two new single subtriangles similar to the original one* (*located at the longest edge of t_0*) *and a block of subtriangles t_1. These subtriangles t_1 either*

1. *are a block of triangles in a basic block* (*t_0 is said to be a Type* 1 *obtuse triangle*)*, or*
2. *a block of similar single triangles, as in Fig.* 7(b) (*t_0 is said to be a Type* 2 *obtuse triangle*).

Proof. Let $\alpha_0 \leqslant \beta_0 \leqslant \gamma_0$ be the angles of the initial obtuse triangle t_0 and let a, b, c be the sides of t_0 respectively opposite to α_0, β_0 and γ_0. For the new non-similar subtriangles generated, we denote by ϵ the opposite angle to MN and σ the opposite angle to CN, see Fig. 7. Since $MN \leqslant CN$ and $\epsilon \leqslant \sigma$ the longest edge of t_1 is either the new edge CM or CN. In the first situation (point 1 of the proposition), triangles t_1 are a basic block sharing edge CM as the longest edge.

In the second case, the largest angle of t_1 is σ (see triangles t_1 in Fig. 7). The new triangles t_1 are not basic blocks (point 2 of the proposition). □

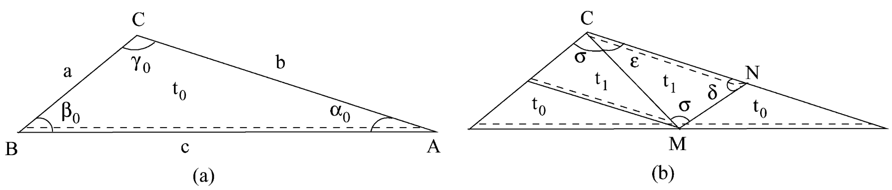

Fig. 7. (a) Type 2 obtuse triangle t_0. (b) 4T-LE subdivision of t_0. Longest-edges are remarked by a dashed line.

It should be noted that the 4T-LE subdivision always produces two new single triangles similar to the original one (located at the longest edge of t_0) and excepting for Type 2 obtuse triangles, a basic block (similar or non-similar to the original one). Moreover, in this scenario, the single triangles generated by the iterative 4T-LE subdivision are those located at the longest edge of the initial triangle, Proposition 4.

The following proposition states the recursive improvement property of the 4T-LE subdivision for obtuse triangles (Rivara and Iribarren, 1996):

Proposition 7. *Let t_0 be any obtuse triangle of smallest angle α_0 and largest angle γ_0. Then*:

(1) *The 4-triangles longest-edge subdivision of t_0 produces one similarly distinct triangle t_1 (of smallest angle α_1 and largest angle γ_1) such that $\alpha_1 \geqslant \alpha_0$, $\gamma_1 \geqslant \gamma_0 - \alpha_1$.*
(2) *The 4-triangles longest-edge subdivision of any obtuse triangle t_0 and its descendants produces a finite sequence of N improved similarly distinct triangles t_i of largest angle γ_i, and smallest angle α_i such that*:
  (a) *t_i is obtuse for $i = 1, 2, \ldots, N-1$, and t_N is non-obtuse,*
  (b) *$\alpha_j \geqslant \alpha_{j-1}$ for $j = 1, 2, \ldots, N$,*
  (c) *$\gamma_j \leqslant \gamma_{j-1} - \alpha_{j-1} \leqslant \gamma_0 - j\alpha_0$ for $j = 1, 2, \ldots, N$,*
  (d) *the subdivision of t_N at most produces a new obtuse triangle t_{N+1}, and at this point no new similarly distinct triangles are generated.*

It is worth noting, in relation to Proposition 7 above, that after a finite number of applications of the 4T-LE subdivision to triangle t_0 and its successors a non-obtuse triangle is obtained. This is a straightforward consequence of statement 2(c) in Proposition 7. Furthermore, after a first non-obtuse triangle is obtained then the iterative application of the 4T-LE subdivision does not generate new non-similar triangles (Plaza et al., 2004).

In view of the previous properties, we have:

Proposition 8. *Let $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a sequence of nested meshes obtained by repeated application of 4T-LE subdivision to the previous mesh. Then, the Block-balancing degree of the meshes tends to 1 as $n \to \infty$.*

Proof. It suffices to prove the result for the case in which the initial mesh τ_0 only contains a single triangle t_0. Then, the number of generated triangles associated with the 4T-LE subdivision at stage n of refinement is:

$$N_n = 4^n \tag{2}$$

First, we prove the proposition for initial right, acute, and the Type 1 obtuse triangles. In this situation, the number of triangles in basic blocks T_n generated at stage n of uniform 4T-LE subdivision satisfies (see Proposition 4):

$$T_n = 4T_{n-1} + 2(N_{n-1} - T_{n-1}) = 2(T_{n-1} + N_{n-1}) \tag{3}$$

with $N_0 = 1$ and $T_0 = 0$.

Solving the recurrence relations (2) and (3) we get:

$$T_n = 4^n - 2^n \tag{4}$$

Therefore,

$$\lim_{n\to\infty} B(\tau_n) = \lim_{n\to\infty} \frac{T_n}{N_n} = 1$$

To complete the proof, we now consider the case of an initial Type 2 obtuse triangle t_0. Table 1 presents the number of distinct types of triangles generated by the 4T-LE iterative refinement of t_0. We denote by t_j^n the number of triangles of similarity class t_j, $j = 0, 1, 2, \ldots, k$, at stage n of refinement. For example, after the second refinement 4 triangles are similar to t_0, 8 triangles similar to t_1 and 4 new triangles similar to t_2.

For each triangle of class t_{j-1} at stage $n-1$ of refinement, there are obtained two triangles of class t_{j-1} and two triangles of class t_j at stage n of refinement. This implies that the number of triangles of similarity class t_j, $j = 0, 1, 2, \ldots, k$, at stage n of refinement t_j^n, satisfy the recurrence relation:

$$t_j^n = 2\left(t_j^{n-1} + t_{j-1}^{n-1}\right), \quad j = 1, 2, 3, \ldots, k \tag{5}$$

Table 1
Triangle evolution in the 4T-LE subdivision

Ref.	0	1	2	3	4	…	k	…	n
t_0	1	2	4	8	16	…	t_0^k	…	t_0^n
t_1		2	8	24	64	…	t_1^k	…	t_1^n
t_2			4	24	96	…	t_2^k	…	t_2^n
t_3				8	64	…	t_3^k	…	t_3^n
t_4					16	…	t_4^k	…	t_4^n
…						…	…	…	…
t_k						…	t_k^k	…	t_k^n

The solution to Eq. (5) with initial condition $t_0^0 = 1$ can be easily expressed in terms of binomial coefficients as follows:

$$t_j^n = 2^n \binom{n}{j} \tag{6}$$

On the other hand, from Proposition 7, the iterative 4T-LE subdivision of any obtuse triangle t_0 produces a finite number of distinct (up to similarity) triangles, t_j^i, $0 < j \leqslant k$. After k refinement stages there will no longer be distinct new generated triangles different from those already generated (see proof of Proposition 6). Therefore, the number of triangles in basic blocks T_n after the k refinement stage with $n > k$ satisfy:

$$T_n \geqslant 2^n \sum_{m=k}^{n} \binom{n}{m}$$

It follows that:

$$1 \geqslant B(\tau_n) \geqslant \frac{2^n \sum_{m=k}^{n} \binom{n}{m}}{2^n \sum_{m=0}^{n} \binom{n}{m}} = \frac{\sum_{m=k}^{n} \binom{n}{m}}{2^n}$$

Taking limits:

$$1 \geqslant \lim_{n\to\infty} B(\tau_n) \geqslant \lim_{n\to\infty} \frac{2^n \sum_{m=k}^{n} \binom{n}{m}}{2^n \sum_{m=0}^{n} \binom{n}{m}}$$

Since

$$\sum_{m=k}^{n} \binom{n}{m} = 2^n - \sum_{m=0}^{k-1} \binom{n}{m} \geqslant 2^n - \binom{n}{k-1} k$$

we have

$$1 \geqslant \lim_{n\to\infty} B(\tau_n) \geqslant \lim_{n\to\infty} \frac{2^n - \binom{n}{k-1} k}{2^n} = 1$$

So, $\lim_{n\to\infty} B(\tau_n) = 1$. □

4.2. The 4T-Similar subdivision case

The 4T-Similar subdivision offers a simpler case in showing that the Block-balancing degree of the refined meshes tends to 1. It should be noted that in this case, the subdivision produces four sub-triangles, two of them have their longest edges on the longest edge of the parent triangle and the other two are a basic block. Next proposition states the main result as already done for the 4T-LE subdivision:

Proposition 9. *Let $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a sequence of nested meshes obtained by repeated application of 4T-Similar subdivision to the previous mesh. Then, the Block-balancing degree of the meshes tends to* 1 *as $n \to \infty$.*

Proof. The number of triangles in basic blocks T_n generated at stage n of uniform *4T Similar* subdivision satisfies:

$$T_n = 4T_{n-1} + 2S_{n-1} \tag{7}$$

where S_{n-1} are the number of single triangles in the stage $(n-1)$.

The number of single triangles S_n is:

$$S_n = 2S_{n-1} \tag{8}$$

as only two triangles remain single from a triangle subdivision.

Solving the recurrence relations (7) and (8) we get:

$$T_n = 4^n T_0 - 2^n S_0 = 4^n(T_0 + S_0) - 2^n S_0 \tag{9}$$

and taking limits to calculate the Block-balancing degree T_n/N_n:

$$\lim_{n\to\infty} B(\tau_n) = \lim_{n\to\infty} \frac{T_n}{N_n} = 1 \qquad \square$$

5. Utility of block-balanced meshes: a tri to quad mesh conversion scheme

To illustrate the applicability of the study carried out in this work a tri to quad mesh conversion strategy similar to that by Velho (2000) is implemented. Our idea to convert to quad meshes, also explored in (Ramaswami et al., 1998), exploits the property of Propositions 8 and 9 to merge triangles in basic blocks and subdivide elements using the Catmull–Clark scheme. Given a triangle mesh τ_i, $i \in [0, n]$ from a sequence of refined meshes $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$, the scheme is as follows:

(1) Compute basic blocks T and single triangle S in mesh τ_i ($S = \overline{T}$).
(2) Merge basic blocks (removing internal diagonals from each basic block in T).
(3) Apply one step of Catmull–Clark subdivision to merged blocks.
(4) Apply one step of Catmull–Clark subdivision to triangles in S.

The set of basic blocks T in step 1 above is obtained by a procedure that firstly sorts the triangle edges by decreasing length and then selects pair of triangles that share their longest edge. It can be noted that this algorithm produces a quad mesh that is always conforming, see an example in Fig. 8.

Remark 2. A pair of triangles in a basic block subdivides into four quads, and a single triangle subdivides into three quads.

According to Remark 2 it is clearly observed that the quad conversion does not result in the same number of elements than in the triangle mesh, since element explosion will be of 2 to 4 and 1 to 3. In this sense, the new quad mesh can be viewed as a refined mesh respecting the original triangle mesh. So this algorithm has to be used with caution if one truly needs a tri to quad conversion with the same amount of elements/vertices through the conversion.

To see the evolution of basic blocks in the 4T-LE iterative refinement we distinguish two different cases: (1) right, acute and Type 1 obtuse triangles and (2) Type 2 obtuse triangles. We perform here a numerical example to observe the behavior of the 4T-LE subdivision. We apply seven stages of uniform 4T-LE refinement to an acute triangle and a Type 2 obtuse triangle. The goal in this experiment is to calculate the number of triangles in basic blocks compared to the amount of single triangles in each stage of the refinement (Table 2). By stage 7 the number of triangles in basic

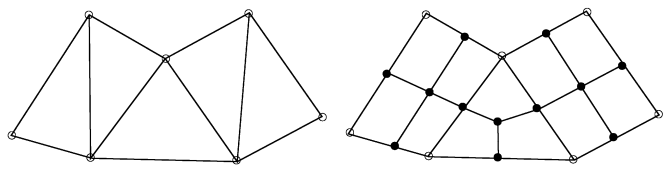

Fig. 8. Tri to quad conversion example.

Table 2
Triangles in basic blocks (T) and Single triangles (S) evolution. Right, Acute, Type 1 obtuse is denoted by (a) and Type 2 obtuse by (b)

Ref. #	T (a)	S (a)	T (b)	S (b)
0	0	1	0	1
1	2	2	0	4
2	12	4	2	14
3	56	8	26	38
4	240	16	162	94
5	992	32	802	222
6	4032	64	3586	510
7	16256	128	15234	1150

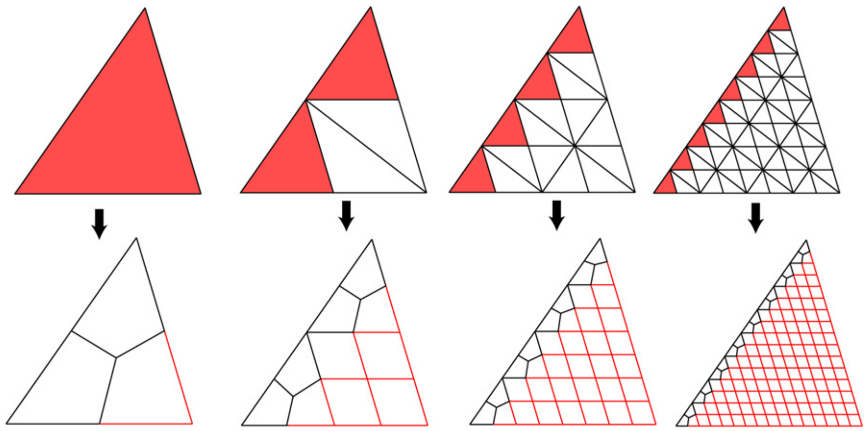

Fig. 9. 4T-LE refinement of an acute triangle (shaded triangles are single triangles an white triangles are basic blocks). Respective quad meshes.

blocks (16,256 and 15,234) is clearly larger than the number of the single triangles (128 and 1,150) and this is in agreement with Proposition 8. In Fig. 9, initial acute triangle with three refined meshes and respective quad meshes are showed. The shaded triangles are single triangles and the other are basic blocks.

To test the 4T-LE refinement technique and the mesh conversion procedure, two triangle mesh samples are considered, consisting of triangulated polygonal surfaces described by points, triangles and their connectivity. One stage of 4T-LE refinement is applied to initial meshes with 1680 (snail mesh) and 1000 (cup mesh) triangles, see initial and refined meshes in Fig. 10(a) and (c). The respective quad meshes are presented in Fig. 10(b) and (d). It should be noted that we are not dealing here with smooth surfaces as those produced by known methods like Velho and Zorin (2001). Moreover, surface parameterization is not achieved in the examples. Therefore, in these test examples, the original polygonal surface is preserved when iterative refinement is applied.

5.1. Mesh quality progress in the 4T-LE iterative refinement

The number of single triangles in the initial mesh determines the number of single triangles in successive refinement stages. The minimum is held for the 4T-Similar subdivision. For example, if an initial mesh contains S single triangles and 4T-Similar subdivision is used, the number of remaining single triangles at refinement stage i will be $2^i S$. In the 4T-LE subdivision, the number of single triangles might be greater as any Type 2 obtuse triangle may produce up to four single triangles in any refinement stage. However, 4T-LE is a valuable option since triangle quality usually improves (Plaza et al., 2004; Rivara and Iribarren, 1996; Rosenberg and Stenger, 1975). The so-called self-improvement property of the refinement algorithm based on the 4T-LE subdivision has been recently discussed and delimited by studying the number of dissimilar triangles arising from the 4T-LE subdivision of an initial triangle and its successors. The quality of the triangles generated improves but only within certain limits as has been found in (Plaza et al., 2004).

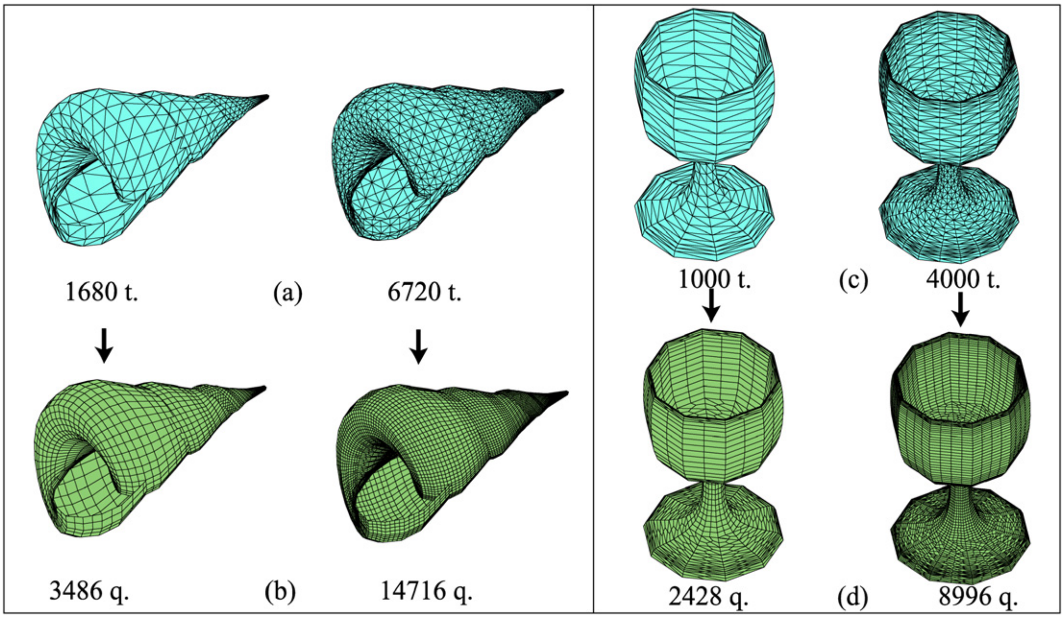

Fig. 10. Tri to quad mesh conversion samples using one stage of 4T-LE triangle refinement.

In the 4T-S scheme the 'parent' triangle is subdivided to a quartet of congruent subtriangles each similar to the parent triangle. Hence an acute-angled parent triangle will guarantee four similar acute subtriangles. Likewise, an obtuse triangle generates four similar obtuse subtriangles under 4T-S subdivision. In the 4T-LE scheme the two subtriangles with edges coincident with the longest edge of the parent are similar to the parent and the remaining pair of triangles form a similar pair that in general, are not similar to the parent triangle. Note that 4TLE subdivision of an equilateral triangle yields two equilateral triangles and two obtuse triangles. Obtuse parent triangles may yield pairs of acute and obtuse triangles and repeated recursive subdivision of obtuse triangles under 4TLE will yield meshes that always contain some obtuse subtriangles. However, it has been proven in (Rivara and Iribarren, 1996) that the iterative subdivision of obtuse triangles systematically improves the triangles (while they remain obtuse) in the following sense: the sequence of smallest angles monotonically increases while the sequence of largest angles monotonically decreases in an amount (at least) equal to the smallest angle of each iteration. In addition, bounds on the values of the angles obtained and the exact number of dissimilar triangles have been given in (Plaza et al., 2004).

To observe the evolution of basic blocks and the quality of the meshes (reporting minimum and maximum angles of triangles) we perform an experiment starting with a bad-shaped mesh, named for convenience, *Pentagonal mesh* with 125 triangles, Fig. 11. To this starting mesh it is applied five stages of uniform refinements using 4T-LE and 4T-S triangle subdivisions. Tables 3 and 4 respectively report number of basic blocks (T) and average of angles (min and max) using these two subdivision schemes until refinement stage five is achieved with 128,000 triangles. It can be noted that 4T-S generates more basic blocks than 4T-LE. However, the largest angles, in average, are improved using 4T-LE subdivision as its average of 120.49° in the starting mesh is decreased to 100.21° in the last refined mesh. Although this improvement is also seen for the average of minimum angles, this is of less importance and can be insignificant in many other cases. In view of that, it seems reasonable to use 4T-Similar subdivision when a faster conversion to quadrangles is desired. However, to improve mesh quality, 4T-LE approach will be a better choice.

6. Conclusions

In this paper we have studied and delimited some properties of two well-known four triangle subdivisions that are useful, for example, in tri to quad mesh conversion algorithms. The main property states that the triangle blocks sharing a common longest edge (basic blocks) tend to cover the area of the mesh as the number of refinement applied tends to infinity, obtaining, so, almost block-balanced meshes. We explain and prove the property for the Four Triangle

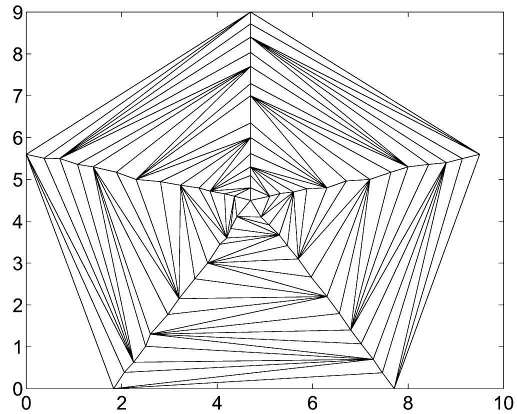

Fig. 11. Mesh in a pentagonal domain.

Table 3
4T-LE iterative refinement in the Pentagonal mesh. Basic blocks (T) and angles average

Triangles	T	μ (Min angles)	μ (Max angles)
125	0	9.1104	120.4942
500	246	9.9262	116.6421
2000	1088	10.9778	112.1217
8000	4778	11.7858	108.2671
32000	21240	12.3874	105.0815
128000	103970	13.7812	100.2149

Table 4
4T-S iterative refinement in the Pentagonal mesh. Basic blocks (T) and angles average

Triangles	T	μ (Min angles)	μ (Max angles)
125	0	9.1104	120.4942
500	250	9.1104	120.4942
2000	1500	9.1104	120.4942
8000	7000	9.1104	120.4942
32000	24240	9.1104	120.4942
128000	112890	9.1104	120.4942

based on the Longest Edge (4T-LE) and the Four Triangle Similar (4T-S) schemes. Although this behavior may be expected in practice, we show theoretical proofs that demonstrate the property.

We point out some other properties of the four triangles longest edge subdivision and the four triangles similar subdivision regarding mesh quality and adjacency relationships, and introduce the block-balancing degree of a mesh as a parameter to measure the amount of basic blocks in a mesh.

We illustrate the reported theory in the paper providing a scheme to convert a tri to quad mesh similar to that by Velho (2000). Our method here offers an 'on the fly' tri to quad conversion mainly adequate for such applications that need simultaneously tri/quad meshes under an iterative refinement procedure. However, for the case of a mesh conversion where no refinement is performed, one should apply other existing methods as proposed in (Ramaswami et al., 1998).

Acknowledgements

This research has been partially supported by Project UNI-2003/16 of the University of Las Palmas de Gran Canaria and by Project PI2003/35 of the Gobierno de Canarias. CYCIT Project number MTM2005-08441-C02-02 from Ministerio de Educación y Ciencia of Spain.

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