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On the Fibonacci k-numbers

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Abstract

We introduce a general Fibonacci sequence that generalizes, between others, both the classic Fibonacci sequence and the Pell sequence. These general kth Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ were found by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge (4TLE) partition. Many properties of these numbers are deduce directly from elementary matrix algebra.

1. Introduction

In the present days there is a huge interest of modern science in the application of the Golden Section and Fibonacci numbers [1–19]. The Fibonacci numbers F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \dots\}$ wherein each term is the sum of the two preceding terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. On the other hand the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\tau = \frac{1+\sqrt{5}}{2}$, which appears in modern research in many fields from Architecture, Nature and Art [20–30] to physics of the high energy particles [31–33] or theoretical physics [34–41].

As an example of the ubiquity of the Golden Mean in geometry we can think of the ratio between the length of a diagonal and a side of a regular pentagon. The paper presented here was originated for the astonishing presence of the Golden Section in a recursive partition of triangles in the context of the finite element method and triangular refinements.

1.1. Grid generation and triangles

Grid generation and, in particular, the construction of 'quality' grids is a major issue in both geometric modeling and engineering analysis [42–46]. Many of these methods employ forms of local and global triangle subdivision and seek to maintain well shaped triangles. The four-triangle longest-edge (4TLE) partition is constructed by joining the midpoint of the longest-edge to the opposite vertex and to the midpoints of the two remaining edges [47,48]. The two subtriangles with edges coincident with the longest-edge of the parent are similar to the parent. The remaining

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two subtriangles form a similar pair that, in general, are not similar to the parent triangle. We refer to such new triangle shapes as 'dissimilar' to those preceding. The iterative partition of obtuse triangles systematically improves the triangles in the sense that the sequence of smallest angles monotonically increases, while the sequence of largest angles monotonically decreases in an amount (at least) equal to the smallest angle of each iteration [44,48].

In this paper, we show the relation between the 4TLE partition and the Fibonacci numbers, as another example of the relation between geometry and numbers. The use of the concept of antecedent of a (normalized) triangle is used to deduce a pair of complex variable functions. These functions, in matrix form, allow us to directly and in an easy way, present many of the basic properties of some of the best known recursive integer sequences, like the Fibonacci numbers and the Pell numbers.

2. Normalized triangles, antecedents and complex valued functions

Since we were interested in the shape of the triangles, each triangle is scaled to have the longest-edge of unit length. In this form, each triangle is represented for the three vertices: $(0,0)$, $(1,0)$ and $z=(x,y)$. Since the two first vertices are the extreme points of the longest-edge, the third vertex is located inside two bounding exterior circular arcs of unit radius, as shown in Fig. 1. In the following, for any triangle t, the edges and angles will be respectively denoted in decreasing order $r_1 \geqslant r_2 \geqslant r_3$, and $\gamma \geqslant \beta \geqslant \alpha$.

Definition 1. The longest-edge (LE) partition of a triangle t_0 is obtained by joining the midpoint of the longest-edge of t_0 with the opposite vertex (Fig. 2(a)). The four-triangle longest-edge (4TLE) partition is obtained by joining the midpoint of the longest-edge to the opposite vertex and to the midpoints of the two remaining edges (see Fig. 2(b)).

In the 4TLE scheme, subdivision leads to subtriangles that are similar to some previous parent triangles in the refinement tree so generated. Other subtriangles may result that are not in such similarity classes yet and we refer to these as new dissimilar triangles. We define the class $\mathscr{C}_n$ as the set of triangles for which the application of the 4TLE partition produces exactly n dissimilar triangles.

Let us begin by describing a Monte Carlo computational experiment used to visually distinguish the classes of triangles by the number of dissimilar triangles generated by the 4TLE partition. We proceed as follows: (1) Select a point within the mapping domain comprised by the horizontal segment and by the two bounding exterior circular arcs. This point (x,y) defines the apex of a target triangle. (2) For this selected triangle, 4TLE refinement is successively applied as long as a new dissimilar triangle appears. This means that we recursively apply 4TLE and stop when the shapes of new generated triangles are the same as those already generated in previous refinement steps. (3) The number of such refinements to reach termination defines the number of dissimilar triangles associated with the initial triangle and this numerical value is assigned to the initial point (x,y) chosen. (4) This process is progressively applied to a large sample of triangles (points) uniformly distributed over the domain. (5) Finally, we graph the respective values of dissimilar triangles in a corresponding color map to obtain the result in Fig. 3.

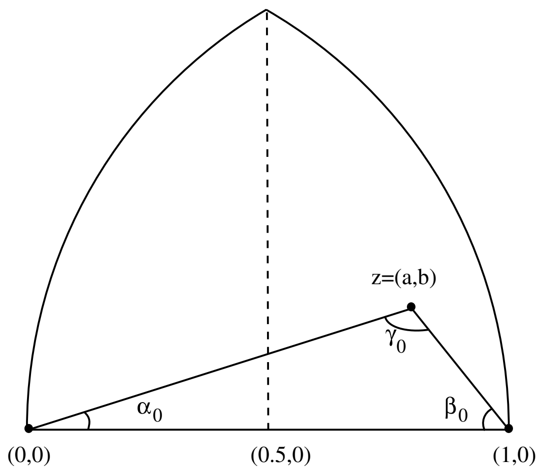

Fig. 1. Diagram for representing shape triangles.

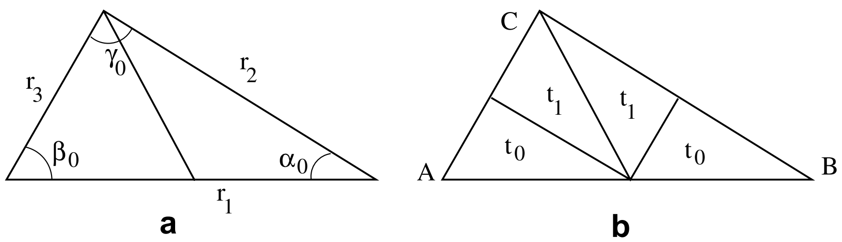

Fig. 2. (a) LE partition of triangle t_0, (b) 4TLE partition of triangle t_0.

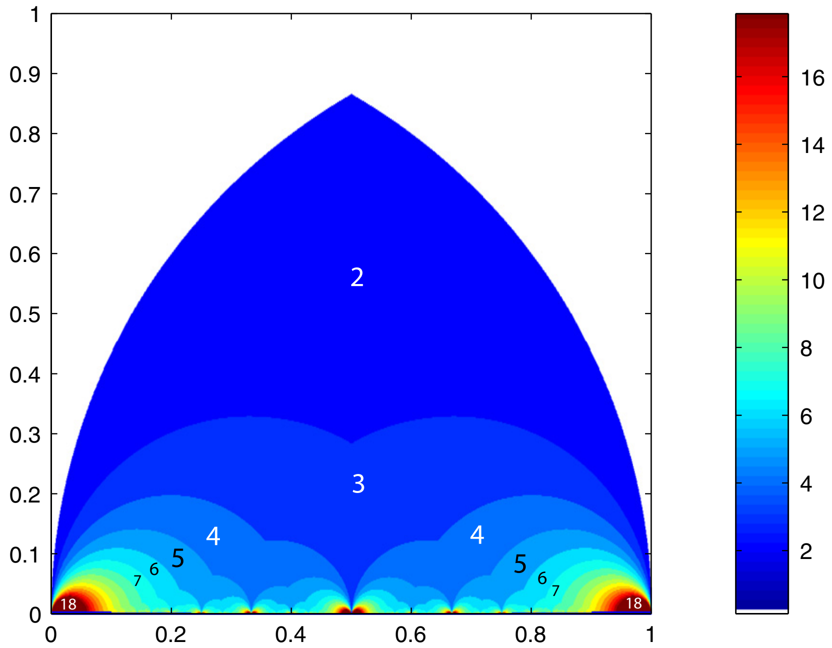

Fig. 3. Subregions for dissimilar triangle classes generated by Monte Carlo simulation.

Definition 2 (4*TLE left and right antecedents*). A given triangle t_{n+1}, has two (different) triangles t_n, denoted here as left and right antecedents, whose 4TLE partition produces triangle t_{n+1}.

As an example, triangle t_{n+1} in the diagram with vertices $(0,0)$, $(1,0)$ and z in Fig. 4(a), has left antecedent t_n with vertices z, $(0,0)$, and $z+1$ in Fig. 4(b), and right antecedent t_n with vertices z, $(1,0)$, and $z-1$ in Fig. 4(c).

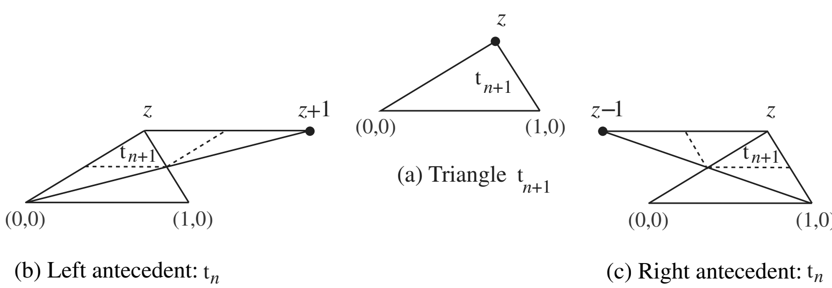

Fig. 4. Two antecedents for the 4TLE partition of triangle t_n.

Theorem 3. *The relation between the apex of a given triangle z in the right half of the diagram and the apices of its left and right antecedents may be mathematically expressed by the maps* $f_L(z) = \frac{1}{\bar{z}+1}$, *and* $f_R(z) = \frac{1}{2-z}$, *complex z.*

Remark. Notice that for z having $\frac{1}{2} \leqslant \mathrm{Re}(z) \leqslant 1$ then $\mathrm{Re}(f_L(z)) \leqslant \mathrm{Re}(f_R(z))$ (and hence the 'left'/'right' terminology given to these complex functions).

Theorem 4. *The class separators determined experimentally in Fig. 3 may be generated mathematically as a recursive composition of left and right maps* $f_L(z)$, *and* $f_R(z)$.

Function f_R is a Moebius transformation (also homography or fractional linear transformation) [49,50], while function f_L is an anti-homography. Both may be considered as maps of the extended plane $\overline{\mathbb{C}}$ into itself. f_R is a conformal map, and hence it preserves angles in magnitude and direction, and straight lines and circles are transformed into straight lines and circles. On the other hand, f_L is not conformal, but angles are preserved in magnitude and reversed in direction, as the complex conjugation. Also f_L takes circles to circles (straight lines count as circles of infinite radius).

In general a Moebius transformation $h(z) = \frac{az+b}{cz+d}$ is defined by the matrix: $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose elements are constant complex numbers and its determinant is not null to avoid the constant transformation. In similar way, an anti-homography presents the form $h(\bar{z}) = \frac{a\bar{z}+b}{c\bar{z}+d}$ and also has the same associated complex matrix.

In our case, let $R = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, and $L = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ be the associated matrices to functions f_R and f_L, respectively. The composition of two of such functions has as associated matrix the product of the matrices associated to the two initial transformations. Similarly, any particular combination of transformations f_R and f_L is determined by the product of the corresponding matrices in the same order. For instance, transformation $(f_R \circ f_L(z)) = f_R(f_L(z)) = f_R(\frac{1}{\bar{z}+1}) = \frac{1}{2-\frac{1}{\bar{z}+1}} = \frac{\bar{z}+1}{2\bar{z}+1}$ could be given more easily by the matrix product $R \cdot L = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. Therefore, from now on, we will substitute the use of the transformations f_R and f_L by the use of the associated matrices R and L.

Let us find the product $R^{k-1}L$ associated to the composition $f_R^{k-1} \circ f_L$ which will be used below. It is easy to prove that $R^{k-1} = \begin{pmatrix} -k+2 & k-1 \\ -k+1 & k \end{pmatrix}$ for all $k \geqslant 1$, and so $R^{k-1} \cdot L = \begin{pmatrix} -k+2 & k-1 \\ -k+1 & k \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix}$.

3. *k*-Fibonacci numbers

In this section, a new generalization of the Fibonacci numbers is introduced. It should be noted that the recurrence formula of these numbers depend on one integral parameter instead of two parameters. We shall show that these numbers are related with the complex valued functions given above, and then, in some sense, with the 4TLE partition of normalized triangles.

Definition 5. For any integer number $k \geqslant 1$, the kth Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for } n \geqslant 1.$$

Particular cases of the previous definition are:

- If $k = 1$, the classic Fibonacci sequence is obtained:
  $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$:
  $\{F_n\}_{n\in\mathbf{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$.
- If $k = 2$, the classic Pell sequence appears:
  $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geqslant 1$:
  $\{P_n\}_{n\in\mathbf{N}} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}$.
- If $k = 3$, the following sequence appears:
  $H_0 = 0$, $H_1 = 1$, and $H_{n+1} = 3H_n + H_{n-1}$ for $n \geqslant 1$:
  $\{H_n\}_{n\in\mathbf{N}} = \{0, 1, 3, 10, 33, 109, \ldots\}$.

The relation between matrix $R^{k-1} \cdot L$ and the kth Fibonacci sequence is given by the following proposition.

Proposition 6. *For any integer* $n \geqslant 1$ *holds*:

$$(R^{k-1}\cdot L)^n = \begin{pmatrix} F_{k,n+1}-F_{k,n} & F_{k,n} \\ F_{k,n+1}-F_{k,n-1} & F_{k,n}+F_{k,n-1} \end{pmatrix}. \tag{1}$$

Proof (*By induction*). For $n=1$:

$$R^{k-1}\cdot L = \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix} = \begin{pmatrix} F_{k,2}-F_{k,1} & F_{k,1} \\ F_{k,2}-F_{k,0} & F_{k,1}+F_{k,0} \end{pmatrix}$$

since $F_{k,0}=0$, $F_{k,1}=1$, and $F_{k,2}=k$.

Let us suppose that the formula is true for $n-1$:

$$(R^{k-1}\cdot L)^{n-1} = \begin{pmatrix} F_{k,n}-F_{k,n-1} & F_{k,n-1} \\ F_{k,n}-F_{k,n-2} & F_{k,n-1}+F_{k,n-2} \end{pmatrix}.$$

Then,

$$(R^{k-1}\cdot L)^n = (R^{k-1}\cdot L)^{n-1}(R^{k-1}\cdot L) = \begin{pmatrix} F_{k,n}-F_{k,n-1} & F_{k,n-1} \\ F_{k,n}-F_{k,n-2} & F_{k,n-1}+F_{k,n-2} \end{pmatrix}\cdot\begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (k-1)F_{k,n}+F_{k,n-1} & F_{k,n} \\ kF_{k,n} & F_{k,n}+F_{k,n-1} \end{pmatrix} = \begin{pmatrix} F_{k,n+1}-F_{k,n} & F_{k,n} \\ F_{k,n+1}-F_{k,n-1} & F_{k,n}+F_{k,n-1} \end{pmatrix}. \qquad \square$$

Particular cases are:

- If $k=1$, the classic Fibonacci sequence is obtained: $F_0=0$, $F_1=1$, and $F_{n+1}=F_n+F_{n-1}$ for $n\geqslant 1$, so $L^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$. This formula shows the relation between the function f_L and the classic Fibonacci sequence. Note that matrix L^n is similar to the nth power of the Fibonacci Q matrix defined by $Q=\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}$, from where $Q^n=\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ [51].
- If $k=2$, we get the classic Pell sequence: $P_0=0$, $P_1=1$, and $P_{n+1}=2P_n+P_{n-1}$ for $n\geqslant 1$, and then: $(R\cdot L)^n = \begin{pmatrix} P_n+P_{n-1} & P_n \\ 2P_n & P_n+P_{n-1} \end{pmatrix}$.
- If $k=3$, sequence $\{H_n\}_{n\in\mathbf{N}}$ is obtained, with $H_0=0$, $H_1=1$, and $H_{n+1}=3H_n+H_{n-1}$, so: $(R^2\cdot L)^n = \begin{pmatrix} 2H_n+H_{n-1} & H_n \\ 3H_n & H_n+H_{n-1} \end{pmatrix}$.

4. Properties from the determinant of matrix $(R^{k-1}\cdot L)^n$

For the shake of clarity we note in the sequel by T the matrix $R^{k-1}\cdot L$. In this section, we shall study some properties for the kth Fibonacci sequences which are directly obtained from the determinant of matrices $T^n=(R^{k-1}\cdot L)^n$, that is, from the associated matrices to transformations f_R and f_L.

Proposition 7 (*Catalan identity*). $F_{k,n+r+1}F_{k,n+r-1}-F^2_{k,n+r}=(-1)^{n+r}$.

Proof. If in Proposition 6 n is changed by $n+r$, the following matrix is obtained:

$$(R^{k-1}\cdot L)^{n+r} = \begin{pmatrix} F_{k,n+r+1}-F_{k,n+r} & F_{k,n+r} \\ F_{k,n+r+1}-F_{k,n+r-1} & F_{k,n+r}+F_{k,n+r-1} \end{pmatrix}$$

and $|(R^{k-1}\cdot L)^{n+r}| = F_{k,n+r+1}F_{k,n+r-1}-F^2_{k,n+r}$. Since $|R|=1$ and $|L|=-1$, we have $|(R^{k-1}\cdot L)^{n+r}|=(-1)^{n+r}$ from where the identity is obtained. $\square$

Particular cases are:

- If $k=1$ and $r=0$, the Cassini's identity or Simson formula for the classic Fibonacci sequence appears: $F_{n+1}F_{n-1}-F_n^2=(-1)^n$.

- If $k = 2$ and $r = 0$, for the Pell sequence it is obtained: $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$.
- If $k = 3$ and $r = 0$, for the sequence $\{H_n\}$, we have: $H_{n+1}H_{n-1} - H_n^2 = (-1)^n$.

The relation between matrix $R^{k-1} \cdot L$ and the kth Fibonacci sequence is given by the following proposition.

5. Properties by summing up matrices $(R^{k-1} \cdot L)^n$

In this section, we shall show some properties for the sum of the terms of the kth Fibonacci sequences, obtained by summing up the first n matrices $(R^{k-1} \cdot L)^n$.

Proposition 8. $\sum_{i=1}^{n} F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1)$.

Proof. Note that the term a_{12} in matrix $T^n = (R^{k-1} \cdot L)^n$ is precisely $F_{k,n}$. Let S_n be the sum of the first n matrices $T^j = (R^{k-1} \cdot L)^j$. That is, $S_n = T + T^2 + \cdots + T^n$. The argument here is the same that used in the proof of the sum of the n first terms of a geometric numerical progression:

Since $S_nT = T^2 + T^3 + \cdots + T^n + T^{n+1}$, then $S_n(T - I_2) = T^{n+1} - T$, where I_2 is the 2×2 unit matrix. And, therefore, $S_n = (T^{n+1} - T)(T - I_2)^{-1}$.

Note, now, that

$$T^{n+1} - T = \begin{pmatrix} F_{k,n+2} - F_{k,n+1} & F_{k,n+1} \\ F_{k,n+2} - F_{k,n} & F_{k,n+1} + F_{k,n} \end{pmatrix} - \begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix} = \begin{pmatrix} F_{k,n+2} - F_{k,n+1} - k + 1 & F_{k,n+1} - 1 \\ F_{k,n+2} - F_{k,n} - k & F_{k,n+1} + F_{k,n} - 1 \end{pmatrix}.$$

On the other hand,

$$T - I_2 = \begin{pmatrix} k-2 & 1 \\ k & 0 \end{pmatrix} \Rightarrow (T - I_2)^{-1} = \frac{1}{k}\begin{pmatrix} 0 & 1 \\ k & 2-k \end{pmatrix}.$$

Therefore,

$$S_n = \frac{1}{k}\begin{pmatrix} F_{k,n+2} - F_{k,n+1} - k + 1 & F_{k,n+1} - 1 \\ F_{k,n+2} - F_{k,n} - k & F_{k,n+1} + F_{k,n} - 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ k & 2-k \end{pmatrix}$$

and, finally, by obtaining the term a_{12} of the previous product, since this term is at the same time $\sum_{i=1}^{n} F_{k,i}$, we get the result. □

Particular cases:

- If $k = 1$, for the classic Fibonacci sequence, we obtain:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

- If $k = 2$, for the Pell sequence we have

$$\sum_{i=1}^{n} P_i = \frac{1}{2}(P_{n+1} + P_n - 1).$$

- If $k = 3$, the sum of the first elements of the sequence $\{H_n\}$ is

$$\sum_{i=1}^{n} H_i = \frac{1}{3}(H_{n+1} + H_n - 1).$$

By summing up the first n even terms of the kth Fibonacci sequence we obtain

Proposition 9. $\sum_{i=1}^{n} F_{k,2i} = \frac{1}{k}(F_{k,2n+1} - 1)$.

Proof. The proof is similar to the proof of Proposition 8, and we only show an outline of it. The sum is $S_{2n} = T^2 + T^4 + \cdots + T^{2n}$ where $T = R^{k-1} \cdot L$. By multiplying by T^2 and, after some algebra, we get: $S_{2n} = (T^{2n+2} - T^2)(T^2 - I_2)^{-1}$.

Note that $(T^2 - I_2)^{-1} = \frac{-1}{k}\begin{pmatrix} 1 & -1 \\ -k & k-1 \end{pmatrix}$ and since, the terms a_{12} of both sides are equal, the formula is obtained. □

Particular cases:

- If $k = 1$, for the classic Fibonacci sequence, we obtain:

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1.$$

- If $k = 2$, for the Pell sequence we have

$$\sum_{i=1}^{n} P_{2i} = \frac{1}{2}(P_{2n+1} - 1).$$

- If $k = 3$, the sum of the first even elements of sequence $\{H_n\}$ is

$$\sum_{i=1}^{n} H_{2i} = \frac{1}{3}(H_{2n+1} - 1).$$

Now, considering Propositions 8 and 9 it is rightly obtained the sum of the first odd terms of the kth Fibonacci sequence:

Corollary 10. $\sum_{i=0}^{n} F_{k,2i+1} = \frac{1}{k} F_{k,2n+2}$.

Particular cases:

- If $k = 1$, for the classic Fibonacci sequence, we obtain:

$$\sum_{i=0}^{n} F_{2i+1} = F_{2n+2}.$$

- If $k = 2$, for the Pell sequence we have

$$\sum_{i=0}^{n} P_{2i+1} = \frac{1}{2} P_{2n+2}.$$

- If $k = 3$, the sum of the first odd elements of sequence $\{H_n\}$ is

$$\sum_{i=0}^{n} H_{2i+1} = \frac{1}{3} H_{2n+2}.$$

In a similar way, many formulas for partial sums of term of the kth Fibonacci sequence may be obtained and particularized for different values of k. For example:

Corollary 11. $\sum_{i=0}^{n} F_{k,4i+1} = \frac{1}{k} F_{k,2n+1} F_{k,2n+2}$.

Let p be a non-null real number. Next Proposition gives us the value for the sum of the first kth Fibonacci numbers with weights p^{-i}:

Proposition 12. *For each non-vanishing real number* p:

$$\sum_{j=1}^{n} \frac{F_{k,j}}{p^j} = \frac{-p}{p^2 - kp - 1}\left[\frac{1}{p^{n+1}}(pF_{k,n+1} + F_{k,n}) - 1\right]. \tag{2}$$

Proof. The proof is similar to those given above, but now considering the matrix $S_n = \sum_{j=1}^{n}\left(\frac{1}{p} R^{k-1} \cdot L\right)^j$. □

Eq. (2) is known as Livio's formula [2]. It should be noted that the denominator of the right-hand side of Livio's formula is precisely the characteristic kth Fibonacci polynomial.

Now, and also by using elementary matrix algebra, we will obtain a closed expression for $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{F_{k,j}}{p^j}$.

Proposition 13. *For each real number p, such that* $p > \frac{k+\sqrt{k^2-4}}{2}$:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{F_{k,j}}{p^j} = \sum_{j=1}^{\infty} \frac{F_{k,j}}{p^j} = \frac{p}{p^2 - kp - 1}. \tag{3}$$

Proof. The proof is based in the so-called Binet's formula for the kth Fibonacci sequence:

Let $r_1 = \frac{k+\sqrt{k^2-4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2-4}}{2}$, where r_1 and r_2 are the fundamental roots of the kth Fibonacci sequence. Then, $F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$.

As a consequence, if $p > r_1$, then $\lim_{n\to\infty} \frac{F_{k,n}}{p^n} = \lim_{n\to\infty} \frac{\left(\frac{r_1}{p}\right)^n - \left(\frac{r_2}{p}\right)^n}{r_1 - r_2} = 0$. And, therefore, $\lim_{n\to\infty}\sum_{j=1}^{n} \frac{F_{k,j}}{p^j} = \frac{p}{p^2-kp-1}$. □

Particular cases:

- If $k = 1$, for the classic Fibonacci sequence is obtained: $\sum_{j=1}^{\infty} \frac{F_j}{p^j} = \frac{p}{p^2-p-1}$, which for $p = 10$ gives: $\sum_{j=1}^{\infty} \frac{F_j}{10^j} = \frac{10}{89}$ [2].
- If $k = 2$, for the classic Pell sequence appears: $\sum_{j=1}^{\infty} \frac{P_j}{p^j} = \frac{p}{p^2-2p-1}$, which for $p = 10$ gives: $\sum_{j=1}^{\infty} \frac{P_j}{10^j} = \frac{10}{79}$.
- If $k = 3$, for sequence $\{H_n\}$ results: $\sum_{j=1}^{\infty} \frac{H_j}{p^j} = \frac{p}{p^2-3p-1}$, which for $p = 10$ gives: $\sum_{j=1}^{\infty} \frac{H_j}{10^j} = \frac{10}{69}$.

6. Properties from the product of matrices $(R^{k-1} \cdot L)^n$

In this section, we shall prove some interesting properties of the kth Fibonacci sequences which may be easily deduced from the product of matrices of the form $(R^{k-1} \cdot L)^n$. The first property is called convolution product:

Proposition 14.

$$F_{k,n+m} = F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1}. \tag{4}$$

Proof. Given the matrices $(R^{k-1} \cdot L)^n$, $(R^{k-1} \cdot L)^m$ as Eq. (1), and considering the term a_{12} of the product $(R^{k-1} \cdot L)^n \times (R^{k-1} \cdot L)^m$, which is equal to the term a_{12} of matrix $(R^{k-1} \cdot L)^{n+m}$ we get the result. □

Particular cases:

- If $k = 1$, for the classic Fibonacci sequence is obtained: $F_{n+m} = F_{n+1}F_m + F_nF_{m-1}$ (Honsberger formula [2]).
- If $k = 2$, for the classic Pell sequence appears: $P_{n+m} = P_{n+1}P_m + P_nP_{m-1}$.

Eq. (4) may be particularized in many ways. For example, if $m = n$ we get: $F_{k,2n} = (F_{k,n+1} + F_{k,n-1})F_{k,n} = (F_{k,n+1} + F_{k,n-1})\frac{F_{k,n+1}-F_{k,n-1}}{k}$, and, therefore,

$$F_{k,2n} = \frac{1}{k}(F_{k,n+1}^2 - F_{k,n-1}^2), \tag{5}$$

which may be particularized as follows:

- If $k = 1$, for the classic Fibonacci sequence is obtained: $F_{2n} = F_{n+1}^2 - F_{n-1}^2$.
- If $k = 2$, for the classic Pell results: $P_{2n} = \frac{1}{2}(P_{n+1}^2 - P_{n-1}^2)$.

On the other hand, if $m = n + 1$ in Eq. (4) we get

$$F_{k,2n+1} = F_{k,n+1}^2 + F_{k,n}^2. \tag{6}$$

By doing $m = 2n$ in Eq. (4) we get

$F_{k,3n} = F_{k,n}(F_{k,n+1}^2 + F_{k,n}^2 + F_{k,n-1}^2) + F_{k,n+1}F_{k,n}F_{k,n-1}$. Now, considering that $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, we get

$$F_{k,3n} = \frac{1}{k}(F_{k,n+1}^3 + kF_{k,n}^3 - F_{k,n-1}^3), \tag{7}$$

which, for $k = 1$ reads: $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$.

Similarly, we can deduce

$$F_{k,4n} = \frac{1}{k}(F^4_{k,n+1} + 2k^2F^4_{k,n} - F^4_{k,n-1}) + 4F^3_{k,n}F_{k,n-1}, \tag{8}$$

which, for $k = 1$ reads: $F_{4n} = F^4_{n+1} + 2F^4_n - F^4_{n-1} + 4F^3_nF_{n-1}$.

Remark. Notice that, if in the matrix product of the beginning of this section we would have considered the term a_{22} instead of the term a_{12}, we would have obtained the following equation

$$F_{k,n+m} + F_{k,n+m-1} = F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m} + F_{k,n}F_{k,m-1} + F_{k,n-1}F_{k,m-1},$$

which, for $k = 1$ may be written as

$$F_{n+m} + F_{n+m-1} = F_{n+2}F_m + F_{n+1}F_{m-1} = F_{n+1}F_{m+1} + F_nF_m.$$

7. Conclusions

New generalized kth Fibonacci sequences have been introduced and studied. Many of the properties of these sequences are proved by simple matrix algebra. This study has been motivated by the arising of two complex valued maps to represent the two antecedents in an specific four-triangle partition.

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