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# The $k$-Fibonacci sequence and the Pascal 2-triangle

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**Abstract**

The general $k$-Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ were found by studying the recursive application of two geometrical transformations used in the well-known 4-triangle longest-edge (4TLE) partition. This sequence generalizes, between others, both the classical Fibonacci sequence and the Pell sequence. In this paper many properties of these numbers are deduced and related with the so-called Pascal 2-triangle.


## 1. Introduction

There is a huge interest of modern science in the application of the Golden Section and Fibonacci numbers [1–20]. The Fibonacci numbers $F_n$ are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \ldots\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. On the other hand the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\tau = \frac{1+\sqrt{5}}{2}$, which appears in modern research, particularly physics of the high energy particles [21–24] or theoretical physics [25–39].

The paper presented here was initially originated for the astonishing presence of the Golden Section in a recursive partition of triangles in the context of the finite element method and triangular refinements. In [40] we showed the relation between the 4-triangle longest-edge (4TLE) partition and the Fibonacci numbers, as another example of the relation between geometry and numbers.

In this paper, we present the $k$-Fibonacci numbers in an explicit way and, by easy arguments, many properties are proven. In particular the $k$-Fibonacci numbers are related with the so-called Pascal 2-triangle.

## 2. The $k$-Fibonacci numbers and properties

In this section, a new generalization of the Fibonacci numbers is introduced. It should be noted that the recurrence formula of these numbers depends on one real parameter. These numbers extend the definition of the $k$-Fibonacci numbers given in [40], where $k$ was a positive integer.

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**Definition 1.** For any positive real number $k$, the $k$-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for } n \geqslant 1 \tag{1}$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1. \tag{2}$$

Note that if $k$ is a real variable $x$ then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \left\{\begin{array}{ll} 1 & \text{if } n = 0, \\ x & \text{if } n = 1, \\ xF_n(x) + F_{n-1}(x) & \text{if } n > 1. \end{array}\right\}$$

Particular cases of the $k$-Fibonacci sequence are

- If $k = 1$, the classical Fibonacci sequence is obtained

  $F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_n - 1 \qquad \text{for } n \geqslant 1:$
  $\{F_n\}_{n\in\mathbf{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}.$

- If $k = 2$, the classical Pell sequence appears

  $P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \qquad \text{for } n \geqslant 1:$
  $\{P_n\}_{n\in\mathbf{N}} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}.$

- If $k = 3$, the following sequence appears:

  $H_0 = 0, \quad H_1 = 1, \quad \text{and} \quad H_{n+1} = 3H_n + H_{n-1} \qquad \text{for } n \geqslant 1:$
  $\{H_n\}_{n\in\mathbf{N}} = \{0, 1, 3, 10, 33, 109, \ldots\}.$

### 2.1. *A first explicit formula for the general term of the k-Fibonacci sequence*

*Binet's formulas* are well known in the *Fibonacci numbers theory* [1,12]. In our case, Binet's formula allows us to express the $k$-Fibonacci number in function of the roots $r_1$ and $r_2$ of the following characteristic equation, associated to the recurrence relation (1):

$$r^2 = kr + 1. \tag{3}$$

**Proposition 2** (Binet's formula). *The nth k-Fibonacci number is given by*

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{4}$$

*where $r_1$, $r_2$ are the roots of the characteristic equation (3), and $r_1 > r_2$.*

**Proof.** The roots of the characteristic equation (3) are $r_1 = \frac{k+\sqrt{k^2+4}}{2}$, and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$.

Note that, since $0 < k$, then

$$r_2 < 0 < r_1 \quad \text{and} \quad |r_2| < |r_1|,$$
$$r_1 + r_2 = k \quad \text{and} \quad r_1 r_2 = -1,$$
$$r_1 - r_2 = \sqrt{k^2 + 4}.$$

Therefore, the general term of the $k$-Fibonacci sequence may be expressed in the form: $F_{k,n} = C_1 r_1^n + C_2 r_2^n$, for some coefficients $C_1$ and $C_2$. Giving to $n$ the values $n = 0$ and $n = 1$ it is obtained $C_1 = \frac{1}{r_1 - r_2} = -C_2$, and therefore $F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$. □

Particular cases are

- If $k = 1$, for the classical Fibonacci sequence, we have: $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. $r_1$ is well-known as *the golden ratio*, say $\tau$, while $r_2$ is usually written as $\varphi$. In this notation the general term for the classical Fibonacci sequence is given by $F_n = \frac{\tau^n - \varphi^n}{\tau - \varphi}$. $\tau$ is also denoted by $\sigma_1$ [18–20].
- If $k = 2$, for the Pell sequence, it is obtained: $r_1 = 1 + \sqrt{2} = \alpha$ and $r_2 = 1 - \sqrt{2} = \beta$. $\alpha$ is known as *the silver ratio*, and also denoted by $\sigma_2$.
- Finally, if $k = 3$, for the sequence $\{H_n\}$, the solutions of the characteristic equation are $r_1 = \frac{3+\sqrt{13}}{2}$ and $r_2 = \frac{3-\sqrt{13}}{2}$. $r_1$ is known as *the bronze ratio* and also denoted by $\sigma_3$ [18–20].

As immediate consequence of Binet's formula given in Eq. (4) two more identities are derived below:

**Proposition 3** (Catalan's identity)

$$F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r}F_{k,r}^2. \tag{5}$$

**Proof.** By using Eq. (4) in the left hand side (LHS) of Eq. (5), and taking into account that $r_1 r_2 = -1$ it is obtained

$$\begin{aligned}
(\text{LHS}) &= \frac{r_1^{n-r} - r_2^{n-r}}{r_1 - r_2} \cdot \frac{r_1^{n+r} - r_2^{n+r}}{r_1 - r_2} - \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)^2 = \frac{r_1^{2n} - r_1^{n-r}r_2^{n+r} - r_1^{n+r}r_2^{n-r} + r_2^{2n} - r_1^{2n} + 2r_1^n r_2^n - r_2^{2n}}{(r_1 - r_2)^2} \\
&= \frac{1}{(r_1 - r_2)^2} \cdot \left(-(r_1 r_2)^n \left(\frac{r_2}{r_1}\right)^r - (r_1 r_2)^n \left(\frac{r_1}{r_2}\right)^r + 2(r_1 r_2)^n\right) = \frac{(-1)^{n+1}}{(r_1 - r_2)^2} \cdot \left(\frac{r_2^{2r} + r_1^{2r}}{(r_1 r_2)^r} - 2\right) \\
&= (-1)^{n+1-r} \frac{(r_1^r - r_2^r)^2}{(r_1 - r_2)^2}
\end{aligned}$$

and, again by Eq. (4), the result is obtained. □

Note that for $r = 1$, Eq. (5) gives Cassini's identity [5] for the $k$-Fibonacci sequence

$$F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n. \tag{6}$$

In a similar way that before the following identity is proven:

**Proposition 4** (d'Ocagne's identity). *If $m > n$, then*

$$F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}. \tag{7}$$

### *2.2. A second formula for the general term of the k-Fibonacci sequence*

Here it will be shown another explicit expression for calculating the general term of the $k$-Fibonacci sequence.

**Proposition 5**

$$F_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i}(k^2+4)^i, \tag{8}$$

*where $\lfloor a \rfloor$ is the floor function of $a$, that is $\lfloor a \rfloor = \sup\{n \in N \mid n \leqslant a\}$ and says the integer part of $a$, for $a \geqslant 0$.*

**Proof.** By using the values of $r_1$ and $r_2$ obtained in Eq. (4), we get

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} = \frac{1}{\sqrt{k^2+4}}\left(\left(\frac{k + \sqrt{k^2+4}}{2}\right)^n - \left(\frac{k - \sqrt{k^2+4}}{2}\right)^n\right)$$

from where, by developing the $n$th powers, it follows:

$$F_{k,n} = \frac{1}{\sqrt{k^2+4}} \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i}\left(\sqrt{k^2+4}\right)^{2i+1}.$$

Finally, by simplifying the last expression, Eq. (8) is proven. □

Particular cases are

- If $k = 1$, for the classical Fibonacci sequence, we have

$$F_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 5^i.$$

- If $k = 2$ we obtain the general term for the Pell sequence

$$P_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 2^{n-1-2i} 8^i,$$

which simplifies as

$$P_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 2^i.$$

- Finally, if $k = 3$, the general term for the sequence $\{H_n\}$ is written as

$$H_n = \left(\frac{3}{2}\right)^{n-1} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} \left(\frac{13}{9}\right)^i.$$

### 2.3. Limit of the quotient of two consecutive terms

An useful property in these sequences is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation

**Proposition 6**

$$\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = r_1. \tag{9}$$

**Proof.** By using Eq. (4)

$$\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = \lim_{n\to\infty} \frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}} = \lim_{n\to\infty} \frac{1 - \left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1} - \left(\frac{r_2}{r_1}\right)^n \frac{1}{r_2}},$$

and taking into account that $\lim_{n\to\infty}(\frac{r_2}{r_1})^n = 0$ since $|r_2| < r_1$, Eq. (9) it is obtained. □

As a consequence, for the classical Fibonacci sequence we have $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \tau$, while for the Pell equation is $\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \alpha$, and for the sequence $\{H_n\}$ is $\lim_{n\to\infty} \frac{H_n}{H_{n-1}} = \sigma_3$.

### 2.4. A third formula for the general term of the k-Fibonacci sequence

**Proposition 7**

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i} \quad \textit{for } n \geqslant 2. \tag{10}$$

**Proof.** By induction:

For $n = 2$ we have $F_{k,2} = \sum_{i=0}^{\lfloor \frac{1}{2} \rfloor} \binom{n-1-i}{i} k^{1-2i} = \binom{1}{0} k^1 = k$, which is true by definition of $F_{k,2}$.

Let us suppose that the formula is true for the terms $F_{k,n-1}$ and $F_{k,n}$. Now, by definition of $F_{k,n+1}$:

$F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, so, by the induction hypothesis,

$$F_{k,n+1} = k\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}k^{n-1-2i} + \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-i}{i}k^{n-2-2i} = k^n + k\cdot\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}k^{n-1-2i} + \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-i}{i}k^{n-2-2i},$$

where if in the last term $i$ is replaced by $i-1$ results

$$F_{k,n+1} = k^n + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}k^{n-2i} + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i-1}k^{n-2i}.$$

And now, having in mind that $\binom{m}{i} + \binom{m}{i-1} = \binom{m+1}{i}$ [41], we obtain

$$F_{k,n+1} = k^n + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\binom{n-1}{i}k^{n-2i}. \qquad \square$$

Particular cases of the $k$-Fibonacci sequence are

- If $k = 1$, the classical Fibonacci sequence is obtained
$$F_n = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i} \quad \text{for } n \geqslant 2.$$
- If $k = 2$, for the classical Pell sequence
$$P_n = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}2^{n-1-2i} \quad \text{for } n \geqslant 2.$$
- If $k = 3$, for the $\{H_n\}$ sequence
$$H_n = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-i}{i}3^{n-1-2i} \quad \text{for } n \geqslant 2.$$

### *2.5. Sum of the first terms of the k-Fibonacci sequence*

Binet's formula (4) allow us to express the sum of the first terms of the $k$-Fibonacci sequence in an easy way.

**Proposition 8** (Sum of first terms). *Let $S_{k,n}$ be the sum of the first $(n+1)$ terms of the k-Fibonacci sequence, that is $S_{k,n} = \sum_{i=0}^{n} F_{k,i}$. Then*

$$S_{k,n} = \frac{1}{k}(F_{k,n+1} + F_{k,n}) - \frac{1}{k}. \tag{11}$$

**Proof.** Considering Eq. (4), $S_{k,n}$ may be written as

$$S_{k,n} = \frac{1}{r_1 - r_2}\sum_{i=0}^{n}(r_1^i - r_2^i). \tag{12}$$

Now, by summing up the geometric partial sums $\sum_{i=0}^{n} r_j^i$ for $j = 1,2$ we obtain

$$S_{k,n} = \frac{1}{r_1 - r_2}\left(\frac{r_1^{n+1} - 1}{r_1 - 1} - \frac{r_2^{n+1} - 1}{r_2 - 1}\right), \tag{13}$$

where $r_1$, $r_2$ are the roots of the characteristic equation of $F_{k,n}$. Now, after some algebra we get

$$S_{k,n} = \frac{-r_1^n - r_1^{n+1} - r_2 + 1 + r_2^n + r_1 + r_2^{n+1} - 1}{(r_1 - r_2)(r_1 - 1)(r_2 - 1)} = \frac{1}{k}\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} + \frac{r_1^n - r_2^n}{r_1 - r_2} - \frac{r_1 - r_2}{r_1 - r_2}\right)$$

from where the result is obtained. $\square$

Particular cases are

- If $k = 1$ for the classical Fibonacci sequence is obtained

  $$S_{1,n} = F_{n+2} - 1$$

  and this sum may also be expressed, by Eq. (8), as

  $$S_{1,n} = -1 + \frac{1}{2^{n+1}} \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+2}{2i-1} 5^i.$$

- If $k = 2$ for the classical Pell sequence

  $$S_{2,n} = \frac{1}{2}(P_{n+1} + P_n) - \frac{1}{2}$$

  and this sum may also be written after some algebra as

  $$S_{2,n} = \frac{1}{2} \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1}{2i} 2^i.$$

### 2.6. Two expressions for the metallic ratios as limits

The metallic ratios or metallic means, following the nomenclature introduced in [18–20] are the positive solutions of the characteristic equation $r^2 = kr + 1$. As we have shown, for $k = 1, 2, 3$ these numbers are respectively $\sigma_1$, $\sigma_2$, $\sigma_3$. Here, two different ways for representing the metallic means are given.

#### 2.6.1. Continued fractions

First, note that from the characteristic equation ($r^2 = kr + 1$) it is immediately obtained $r = k + \frac{1}{r}$, from where by repeated substitutions we have

$$r = k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \cdots}}} = [\bar{k}].$$

Note that the last continued fraction represents the positive root of the characteristic equation, say $r_1$, since all the terms are positive. Besides, for different values of parameter $k$ we have the continued fraction corresponding of some of the most common $k$-Fibonacci sequences. That is, for $k = 1$ and the classical Fibonacci sequence we have

$$\sigma_1 = \tau = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = [\bar{1}].$$

For $k = 2$ and the Pell sequence, it is obtained

$$\sigma_2 = \alpha = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} = [\bar{2}],$$

while for $k = 3$ and the $\{H_n\}$ sequence, it is obtained

$$\sigma_3 = 3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3 + \cdots}}} = [\bar{3}].$$

#### 2.6.2. Nested radicals

Also from the characteristic equation we may write $r = \sqrt{1 + kr}$ and by applying iteratively this equation it can be written

$$r = \sqrt{1 + kr} = \sqrt{1 + k\sqrt{1 + k\sqrt{1 + k\sqrt{1 + k \cdots}}}}$$

and, as in the case of continued fractions, note that this expression corresponds to the positive root of the characteristic equation.

Particular cases for different values of $k$ are:

If $k = 1$ and for the classical Fibonacci sequence, the golden ratio or Golden Mean is written as

$$\sigma_1 = \tau = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}.$$

If $k = 2$ and for the Pell sequence, the silver ratio or silver mean is written as

$$\sigma_2 = \alpha = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \cdots}}}}.$$

Finally, if $k = 3$ and for $\{H_n\}$ sequence, the bronze ratio or bronze mean is written as

$$\sigma_3 = \sqrt{1 + 3\sqrt{1 + 3\sqrt{1 + 3\sqrt{1 + \cdots}}}}.$$

## 3. The Pascal 2-triangle and properties

In this section, the Pascal 2-triangle is presented and studied by means of the $k$-Fibonacci sequences, and some of the elementary properties of this Pascal triangle are proven straightforwardly. As a particular case of the Pascal 2-triangle, the classical Pascal triangle is obtained.

By the definition of the $k$-Fibonacci numbers, the first of them are

$$\begin{aligned} F_{k,1} &= 1, \\ F_{k,2} &= k, \\ F_{k,3} &= k^2 + 1, \\ F_{k,4} &= k^3 + 2k, \\ F_{k,5} &= k^4 + 3k^2 + 1, \\ F_{k,6} &= k^5 + 4k^3 + 3k, \\ F_{k,7} &= k^6 + 5k^4 + 6k^2 + 1. \end{aligned}$$

Note that Eq. (10) also gives $F_{k,n}$ as a polynomial on $k$. For instance, the seventh term of the 4-Fibonacci sequence is $F_{4,7} = 4^6 + 5 \cdot 4^4 + 6 \cdot 4^2 + 1 = 5473$.

Table 1
The Pascal 2-triangle

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | | | | | | | 1 | | | | | | |
| 2. | | | | | | | 1 | | | | | | |
| 3. | | | | | | 1 | | 1 | | | | | |
| 4. | | | | | | 1 | | 2 | | | | | |
| 5. | | | | | 1 | | 3 | | 1 | | | | |
| 6. | | | | | 1 | | 4 | | 3 | | | | |
| 7. | | | | 1 | | 5 | | 6 | | 1 | | | |
| 8. | | | | 1 | | 6 | | 10 | | 4 | | | |
| 9. | | | 1 | | 7 | | 15 | | 10 | | 1 | | |
| 10. | | | 1 | | 8 | | 21 | | 20 | | 5 | | |
| 11. | | 1 | | 9 | | 28 | | 35 | | 15 | | 1 | |
| 12. | | 1 | | 10 | | 36 | | 56 | | 35 | | 6 | |
| 13. | 1 | | 11 | | 45 | | 84 | | 70 | | 21 | | 1 |
| 14. | 1 | | 12 | | 55 | | 120 | | 126 | | 56 | | 7 |

It is worthy to be noted that the coefficients arising in the previous list can be written in triangular position, in such a way that every side of the triangle is double, and for this reason this triangle will be called here *Pascal 2-triangle*. See Table 1.

Note that the numbers belonging to the same row of the Pascal 2-triangle are the coefficients of $F_{k,n}$ as they are expressed in Eq. (10). In we note by $F_{k,n}^{(i)}$ the $i$th coefficient in the expression of $F_{k,n}$ as polynomial on $k$, then

$$F_{k,n}^{(i)} = \binom{n-i}{i-1}. \tag{14}$$

For example, the fourth element in the eleventh row is $F_{k,11}^{(4)} = \binom{11-4}{4-1} = \binom{7}{3} = 35$.

Note, also, that the subsequent numbers into the Pascal 2-triangle can be calculated from the previous rows as following equation establishes:

$$F_{k,n+1}^{(i+1)} = F_{k,n}^{(i+1)} + F_{k,n-1}^{(i)}. \tag{15}$$

For example, the third element in the 10th row is the sum of the third element in the 9th row plus the second term in the 8th row. That is, $15 + 6 = 21$.

Also note that Eq. (15) is a direct application of the well-known property of the binomial coefficients: $\binom{m}{i} + \binom{m}{i-1} = \binom{m+1}{i}$ [41]. It should be noted, that the number on the $m$th diagonal of the Pascal 2-triangle are precisely the coefficients of the Taylor expansion of function

$$g_m(x) = \frac{1}{(1-x)^m} \tag{16}$$

centered at the origin.

### 3.1. A simple explanation of the Pascal 2-triangle

Let us consider the two sets of points in the coordinate axes $X = \{x = (x,0)/x \in N\}$ and $Y = \{y = (0,y)/y \in N\}$. A path between an $x$-point and a $y$-point is the not reversing path in the first quadrant from $x$ to $y$ by horizontal and vertical unit segments. For example, from point $x = (2,0)$ to point $y = (0,1)$ there are three paths: $\{(2,0),(1,0),(0,0),(0,1)\}$, $\{(2,0),(1,0),(1,1),(0,1)\}$, and $\{(2,0),(2,1),(1,1),(0,1)\}$.

Note that the diagonals in the Pascal 2-triangle give the number of such paths between an $x$-point and an $y$-point.

In Table 2, each element is the result of summing up the elements on the immediate preceding row from the left to the element above. For instance, number 35 in the fourth row ($x = 3$) is the result of summing up the elements of the third row: $1 + 3 + 6 + 10 + 15 = 35$. In [41, p. 164] the elements of Table 2 are given in the form $(-1)^i\binom{-n}{i}$. In addition, if in Table 2 each row is moved two positions to the right with respect the preceding row, Table 3 is obtained.

Note that each column of Table 3 is equal to the corresponding diagonal of the Pascal 2-triangle which is also known as the "*deformed*" *Pascal Triangle* [17].

Table 2
Number of paths between $x$-points and $y$-points

| $x$ | $y$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 |
| 3 | 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 | 165 |
| 4 | 1 | 5 | 15 | 35 | 70 | 126 | 210 | 330 | 495 |
| 5 | 1 | 6 | 21 | 56 | 126 | 252 | 462 | 792 | 1297 |
| 6 | 1 | 7 | 28 | 84 | 210 | 462 | 924 | 1716 | 3013 |
| 7 | 1 | 8 | 36 | 120 | 330 | 792 | 1716 | 3432 | 6445 |
| 8 | 1 | 9 | 45 | 165 | 495 | 1287 | 3003 | 6435 | 12,880 |
| 9 | 1 | 10 | 55 | 220 | 715 | 2002 | 5005 | 11,440 | 24,320 |

Table 3
Rectangular Pascal 2-triangle

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ... |
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ... |
| | | | | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | ... |
| | | | | | | 1 | 4 | 10 | 20 | 35 | 56 | 84 | ... |
| | | | | | | | | 1 | 5 | 15 | 35 | 70 | ... |
| | | | | | | | | | | 1 | 6 | 21 | ... |
| | | | | | | | | | | | | 1 | ... |

### 3.2. Properties for the diagonals of the Pascal 2-triangle

It will be called here *double diagonal* to each of the different lines of couples of adjacent numbers on the Pascal 2-triangle as shown in Table 1, from right to left and from first row to bottom. For example, the third double diagonal is $\{1\text{–}3, 6\text{–}10, 15\text{–}21, 28\text{–}36, \ldots\}$. These lines are also called *anti-diagonals*.

Note that if in the function given in Eq. (16) we substitute $x$ for a small value, for example $x = 10^{-r}$, for $r \in N$, then the quotient $g_n(x) = \frac{1}{(1-10^{-r})^n}$ if written as a decimal number has its integer part equal to 1, and its decimal part can be seen as $r$-uplas showing the first terms of the $n$th anti-diagonal of the Pascal 2-triangle. For example,

$$g_4(10^{-3}) = \frac{1}{(1-10^{-3})^4} = 1.\underbrace{004}\,\underbrace{010}\,\underbrace{020}\,\underbrace{035}\,\underbrace{056}\ldots \to \{1, 4, 10, 20, 35, 56, \ldots\}$$

which are the first terms in the fourth anti-diagonal of the Pascal 2-triangle.

It will be called here *simple diagonal* each line of numbers from left to right and from top to bottom. For example, the third simple diagonal is:

$$\{1, 3, 6, 10, 15, 21, 28, 36, \ldots\}.$$

It is easy to check that the $i$th double diagonal is equal to the same order simple diagonal, and, therefore, we will call them a diagonal (simple) or anti-diagonal (double).

Some remarks are below in order

(a) The second diagonal is the sequence of the integer numbers, the third diagonal correspond to the triangular numbers, the fourth one to the tetrahedral numbers, and so on. So, the $i$th diagonal is for the $(i-1)$-dimensional simplex numbers [41].
(b) Each element of a diagonal results in the sum of the same order element and all the previous elements in the preceding diagonal.
(c) The sum of two consecutive terms of the third diagonal results a perfect square. The proof is quite easy since by Eq. (14) this sum can be written as $F_{k,n}^{(3)} + F_{k,n+1}^{(3)} = \binom{n-3}{2} + \binom{n-2}{2} = \frac{(n-3)(n-4)}{2} + \frac{(n-2)(n-3)}{2} = (n-3)^2$.
(d) The terms in the third diagonal verify that $8F_{k,n}^{(3)} + 1$ is a perfect square: $8F_{k,n}^{(3)} + 1 = (2n-7)^2$.
(e) By subtracting the odd number sequence to the triangular number sequence, the triangular number sequence is obtained again.
(f) Tetrahedral numbers may be obtained from Table 4, in which the first row is for the integer numbers and the following rows are respectively two, three,... times the integer numbers. In Table 4 the sum of the elements in the anti-diagonals gives the tetrahedral numbers: $\{1, 4, 10, 20, 35, \ldots\}$ [41].

Table 4
Successive multiples of integer numbers

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | ... |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | ... |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | ... |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | ... |
| ... | | | | | | | |

Table 5
Classical Pascal triangle

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | 1 | | | | |
| 1 | | | | 1 | | 1 | | | |
| 2 | | | 1 | | 2 | | 1 | | |
| 3 | | 1 | | 3 | | 3 | | 1 | |
| 4 | 1 | | 4 | | 6 | | 4 | | 1 |
| ... | | | | | | | | | |

(g) If we represent by $A_r(n)$ the $n$th element of the $i$th anti-diagonal, then $A_r(n) = \binom{n+r-2}{r-1}$ as can be proven easily by induction on $r$, as follows:

For $r = 1$ is trivial since $A_1(n) = \binom{n-1}{0} = 1$.

Let us suppose the formula is true for any anti-diagonal of order less or equal than $r$. Then:
$A_{r+1}(n) = A_r(1) + A_r(2) + \ldots + A_r(n) = \binom{r-1}{r-1} + \binom{r-1}{r-1} + \cdots + \binom{n+r-2}{r-1} = \binom{r-1}{0} + \binom{r-1}{1} + \cdots + \binom{k+r-2}{n-1} = \sum_{k=0}^{n-1}\binom{k+r-1}{k} = \binom{n+r-1}{r}$, for the summation formula of the binomial coefficients [41].

### 3.3. Properties of the rows of the Pascal 2-triangle

By inspection of Table 1, may be observed some of the properties of the rows in the Pascal 2-triangle:

(a) By summing up and subtracting alternatively all the elements of a common row the following sequence of cycles is obtained $\{1, 1, 0, -1, -1, 0, \ldots\}$.

(b) The sum of the elements of the $n$th row is equal to the corresponding Fibonacci number. That is: $\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} T_n^i = F_n$, and so $\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n-i}{i-1} = F_n$, as it was obtained before.

(c) The sum of the elements of the $(n-1)$th row with the terms of the $(n+1)$th row gives the corresponding Lucas number. That is, $\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} F_{k,n-1}^{(i)} + \sum_{i=0}^{\lfloor\frac{n+2}{2}\rfloor} F_{k,n+1}^{(i)} = L_n$.

Previous expression is another version of the well-know relation between Fibonacci and Lucas numbers: $L_n = F_{n+1} + F_{n-1}$.

(d) Beginning with 1 on the left, and writing down the following number on its right in the immediately lower row, and so on, the classical Pascal triangle is obtained (Table 5).

## 4. Generating functions for the $k$-Fibonacci sequences

In this section, the generating functions for the $k$-Fibonacci sequences are given. As a result, $k$-Fibonacci sequences are seen as the coefficients of the power series of the corresponding generating function.

Let us suppose that the Fibonacci numbers of order $k$ are the coefficients of a potential series centered at the origin, and let us consider the corresponding analytic function $f_k(x)$. The function defined in such a way is called the generating function of the $k$-Fibonacci numbers. So,

$$f_k(x) = F_{k,0} + F_{k,1}x + F_{k,2}x^2 + \cdots + F_{k,n}x^n + \cdots$$

And then,

$$kxf_k(x) = kF_{k,0}x + kF_{k,1}x^2 + kF_{k,2}x^3 + \cdots + kF_{k,n}x^{n+1} + \cdots,$$
$$x^2f_k(x) = F_{k,0}x^2 + F_{k,1}x^3 + F_{k,2}x^4 + \cdots + F_{k,n}x^{n+2} + \cdots$$

From where, since $F_{k,i} = kF_{k,i-1} + F_{k,n-2}$, $F_{k,0} = 0$, and $F_{k,1} = 1$, it is obtained

$$(1 - kx - x^2)f_k(x) = x.$$

So the generating function for the $k$-Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ is $f_k(x) = \frac{x}{1-kx-x^2}$.

Note that by doing the quotient of the generating function a power series, centered at the origin appears: $f_k(x) = x + kx^2 + (k^2 + 1)x^3 + (k^3 + 2k)x^4 + (k^4 + 3k^2 + 1)x^5 + \cdots$, where the coefficients of the $k$ polynomials are precisely those in the Pascal 2-triangle.

## 5. Conclusions

New generalized $k$-Fibonacci sequences have been introduced and studied. Several properties of these numbers are deduced and related with the so-called Pascal 2-triangle. In addition, the generating functions for these $k$-Fibonacci sequences have been given.

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