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The k-Fibonacci hyperbolic functions

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Abstract

An extension of the classical hyperbolic functions is introduced and studied. These new k-Fibonacci hyperbolic functions generalize also the k-Fibonacci sequences, say $\{F_{k,n}\}_{n=0}^{\infty}$, recently found by studying the recursive application of two geometrical transformations onto $\overline{\mathbf{C}} = \mathbf{C} \cup \{+\infty\}$ used in the well-known four-triangle longest-edge (4TLE) partition. In this paper, several properties of these k-Fibonacci hyperbolic functions are studied in an easy way. We finalize with the introduction of some curves and surfaces naturally related with the k-Fibonacci hyperbolic functions.

1. Introduction

One of the simplest and most celebrated integer sequences is the Fibonacci sequence. The Fibonacci sequence is $\{F_n\} = \{0, 1, 1, 2, 3, 5, \ldots\}$ wherein each term is the sum of the two preceding terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. It is interesting to emphasize the fact that the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\phi = \frac{1+\sqrt{5}}{2}$, which appears in modern research in many fields from Architecture [1,2] to physics of the high energy particles [3–6] or theoretical physics [7–13].

Golden Section and Fibonacci numbers have been studied from different points of view in modern science [14–24]. In a recent paper [25] we showed the relation between the four-triangle longest-edge (4TLE) partition and the k-Fibonacci numbers, as another example of the relation between geometry and numerical sequences. Moreover, the k-Fibonacci numbers have been exposed in an explicit way and are related with the so-called Pascal 2-triangle [26].

This paper presents the continuous versions of the k-Fibonacci numbers which are the k-Fibonacci functions. These functions arise naturally from the k-Fibonacci numbers and can be seen as a new type of hyperbolic functions, where the golden ratio $\phi = \sigma_1$, or more generally σ_k for each real number k, plays an analogous role that the number e into the classical hyperbolic functions.

1.1. The k-Fibonacci numbers

For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by [25]:

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for } n \geqslant 1$$

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Particular cases of the k-Fibonacci sequence for $k = 1, 2$ are, respectively, the classical Fibonacci sequence: $F_0 = 0, F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$, and the Pell sequence: $P_0 = 0$, $P_1 = 1, P_{n+1} = 2P_n + P_{n-1}$ for $n \geqslant 1$.

Note that the characteristic equation associated with the recursive definition of the k-Fibonacci numbers is

$$r^2 = kr + 1 \tag{1}$$

Let us consider the geometric sequence $\{1, \sigma, \sigma^2, \ldots, \sigma^n, \ldots\}$ in which σ is the positive root of Eq. (1), that is $\sigma^2 = k \cdot \sigma + 1$. This geometric sequence is a particular case of a k-Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbf{N}}$, in which $F_{k,0} = 1$, $F_{k,1} = \sigma$, $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ *for* $n \geqslant 1$. It should be noted that every geometric sequence can be seen as a k-Fibonacci sequence, for the appropriate value of k, since k and σ are related by $\sigma^2 = k \cdot \sigma + 1$. Note that in this sequence, if $\sigma \neq 1$ the sequence is non-trivial and, as all geometric sequence, verifies that the quotient of two consecutive terms is equal to a constant number called the ratio of the sequence. We will call here the previous sequence by *metallic sequence*, and this name will be justified in the sequel.

One of the most common formulas in the context of the Fibonacci theory is Binet's formula, which for the k-Fibonacci numbers are given by (see [25]):

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

where r_1, r_2 are the roots of the characteristic equation (1) defining $F_{k,n}$.

An useful property in these sequences is that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation (see [25]): $\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = r_1$ where r_1 is the positive root of Eq. (1).

2. New k-Fibonacci hyperbolic functions

As is well-known, the classical hyperbolic functions are defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

On the other hand, recently, the Fibonacci hyperbolic functions have been defined as [19,21,23]:

$$\text{sFh}x = \frac{\phi^{2x} - \phi^{-2x}}{\sqrt{5}} = \frac{\psi^x - \psi^{-x}}{\sqrt{5}}$$
$$\text{cFh}x = \frac{\phi^{(2x+1)} + \phi^{-(2x+1)}}{\sqrt{5}} = \frac{\psi^{x+1/2} + \psi^{-(x+1/2)}}{\sqrt{5}}$$

where sFh and cFh are called, respectively, the Fibonacci hyperbolic sine and cosine, $\phi = \frac{1+\sqrt{5}}{2}$, and $\psi = \phi^2 = 1 + \phi$.

The above functions may be extended to the k-Fibonacci hyperbolic functions in the following way:

$$\text{sF}_k\text{h}(x) = \frac{\sigma_k^{2x} - \sigma_k^{-2x}}{\sqrt{k^2 + 4}}$$
$$\text{cF}_k\text{h}(x) = \frac{\sigma_k^{(2x+1)} + \sigma_k^{-(2x+1)}}{\sqrt{k^2 + 4}}$$

under the condition that σ_k is the positive root of the characteristic equation associated to the k-Fibonacci sequence, that is $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$. Notice that these functions verify the property that, if x is an even number, $x = 2n$ then $\text{sF}_k\text{h}(x) = F_{k,2n}$, while if x is an odd number, $x = 2n + 1$ then $\text{cF}_k\text{h}(x) = F_{k,2n+1}$. Note that $\text{sF}_k\text{h}(x)$ is symmetric with respect to the origin, while the graphic of $\text{cF}_k\text{h}(x)$ presents a symmetry with respect to the axis $x = -\frac{1}{2}$. By doing a change of variable $2x + 1 = t$ in function cF_kh, the symmetry is for the new variable t with respect to the axis $t = 0$. For this reason, from now on, in analogous way followed by Stakhov and Rozin [19,21] the so-called k-Fibonacci hyperbolic sine and cosine functions are, respectively, defined as follows:

$$\text{sF}_k\text{h}(x) = \frac{\sigma_k^x - \sigma_k^{-x}}{\sigma_k + \sigma_k^{-1}}$$
$$\text{cF}_k\text{h}(x) = \frac{\sigma_k^x + \sigma_k^{-x}}{\sigma_k + \sigma_k^{-1}}$$

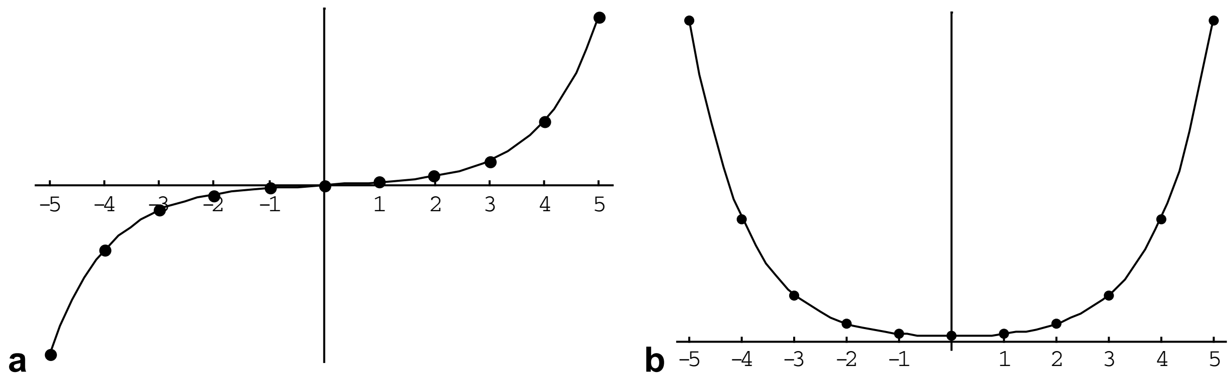

Fig. 1. The Fibonacci hyperbolic functions sFh(x) (left) and cFh(x) (right).

since $\sigma_k + \sigma_k^{-1} = \sqrt{k^2+4}$.

The graphics of these new k-Fibonacci hyperbolic functions are shown in Fig. 1 with the more appealing symmetry with respect to the vertical axis $x = 0$ for the cosine function, and, besides $\mathrm{cF}_k\mathrm{h}(0) = \frac{2}{\sqrt{k^2+4}}$, $\mathrm{cF}_k\mathrm{h}(1) = 1$.

In addition, also by the last definitions, it holds that $\mathrm{sF}_k\mathrm{h}(2n) = F_{k,2n}$ and $\mathrm{cF}_k\mathrm{h}(2n+1) = F_{k,2n+1}$, as it can be easily verified by using $\sigma_k^{-2n} = \left(\frac{-1}{\sigma_k}\right)^{2n}$ and $\sigma_k^{-(2n+1)} = -\left(\frac{-1}{\sigma_k}\right)^{2n+1}$.

The k-Fibonacci hyperbolic functions are related with the classical hyperbolic functions by the following identities:

$$\mathrm{sF}_k\mathrm{h}(x) = \frac{2}{\sigma_k + \sigma_k^{-1}} \sinh(x \ln \sigma_k)$$
$$\mathrm{cF}_k\mathrm{h}(x) = \frac{2}{\sigma_k + \sigma_k^{-1}} \cosh(x \ln \sigma_k)$$

From now on for simplicity we will write σ instead of σ_k.

2.1. Properties of the k-Fibonacci hyperbolic functions

In the sequel we present the main properties of these functions in a similar way in which the analogous properties of the classical hyperbolic functions are usually presented.

Proposition 1 (Pythagorean theorem). *The main property of these functions which it may be called as a version of the Pythagorean Theorem is*

$$[cF_kh(x)]^2 - [sF_kh(x)]^2 = \frac{4}{k^2+4} \tag{2}$$

Notice, that since $k \neq 0$ in the previous identity the right hand side is a positive number less than one.

Proposition 2 (*Sum and difference*).

$$\mathrm{cF}_k\mathrm{h}(x+y) = \frac{\sqrt{k^2+4}}{2}(\mathrm{cF}_k\mathrm{h}(x)\cdot \mathrm{cF}_k\mathrm{h}(y) + \mathrm{sF}_k\mathrm{h}(x)\cdot \mathrm{sF}_k\mathrm{h}(y))$$
$$\mathrm{cF}_k\mathrm{h}(x-y) = \frac{\sqrt{k^2+4}}{2}(\mathrm{cF}_k\mathrm{h}(x)\cdot \mathrm{cF}_k\mathrm{h}(y) - \mathrm{sF}_k\mathrm{h}(x)\cdot \mathrm{sF}_k\mathrm{h}(y))$$
$$\mathrm{sF}_k\mathrm{h}(x+y) = \frac{\sqrt{k^2+4}}{2}(\mathrm{sF}_k\mathrm{h}(x)\cdot \mathrm{cF}_k\mathrm{h}(y) + \mathrm{cF}_k\mathrm{h}(x)\cdot \mathrm{sF}_k\mathrm{h}(y))$$
$$\mathrm{sF}_k\mathrm{h}(x-y) = \frac{\sqrt{k^2+4}}{2}(\mathrm{sF}_k\mathrm{h}(x)\cdot \mathrm{cF}_k\mathrm{h}(y) - \mathrm{cF}_k\mathrm{h}(x)\cdot \mathrm{sF}_k\mathrm{h}(y))$$

Proof. Let us prove the first identity:

$$\mathrm{cF}_k\mathrm{h}(x)\cdot\mathrm{cF}_k\mathrm{h}(y)+\mathrm{sF}_k\mathrm{h}(x)\cdot\mathrm{sF}_k\mathrm{h}(x)=\frac{(\sigma^x+\sigma^{-x})(\sigma^y+\sigma^{-y})+(\sigma^x-\sigma^{-x})(\sigma^y-\sigma^{-y})}{(\sigma+\sigma^{-1})^2}$$
$$=\frac{2}{\sigma+\sigma^{-1}}\cdot\frac{\sigma^{x+y}+\sigma^{-(x+y)}}{\sigma+\sigma^{-1}}=\frac{2}{\sqrt{k^2+4}}\cdot\mathrm{cF}_k\mathrm{h}(x+y)$$

from where the first identity is deduced. □

By doing $y=x$ in the first and third previous formula, we have the following corollary.

Corollary 3 (*Double argument*).

$$\mathrm{cF}_k\mathrm{h}(2x)=\frac{\sqrt{k^2+4}}{2}([\mathrm{cF}_k\mathrm{h}(x)]^2+[\mathrm{sF}_k\mathrm{h}(x)]^2) \tag{3}$$
$$\mathrm{sF}_k\mathrm{h}(2x)=\sqrt{k^2+4}\mathrm{sF}_k\mathrm{h}(x)\cdot\mathrm{cF}_k\mathrm{h}(x) \tag{4}$$

From Eqs. (2) and (4) it is deduced, respectively, by summing up and subtracting:

Corollary 4 (*Half argument*).

$$[\mathrm{cF}_k\mathrm{h}(x)]^2=\frac{1}{\sqrt{k^2+4}}\left(\mathrm{cF}_k\mathrm{h}(2x)+\frac{2}{\sqrt{k^2+4}}\right)$$
$$[\mathrm{sF}_k\mathrm{h}(x)]^2=\frac{1}{\sqrt{k^2+4}}\left(\mathrm{cF}_k\mathrm{h}(2x)-\frac{2}{\sqrt{k^2+4}}\right)$$

Finally, the nth derivatives of these functions verify:

Corollary 5 (nth derivatives).

$$[\mathrm{cF}_k\mathrm{h}(x)]^{(n)}=\left\{\begin{array}{ll}(\ln\sigma)^n\cdot\mathrm{sF}_k\mathrm{h}(x) & \text{for } n=2m+1\\ (\ln\sigma)^n\cdot\mathrm{cF}_k\mathrm{h}(x) & \text{for } n=2m\end{array}\right\}$$
$$[\mathrm{sF}_k\mathrm{h}(x)]^{(n)}=\left\{\begin{array}{ll}(\ln\sigma)^n\cdot\mathrm{cF}_k\mathrm{h}(x) & \text{for } n=2m+1\\ (\ln\sigma)^n\cdot\mathrm{sF}_k\mathrm{h}(x) & \text{for } n=2m\end{array}\right\}$$

It should be noted that in the case $k=0$ the k-Fibonacci sequence would be the sequence $\{0,1,0,1,0,1,\ldots\}$ with $\sigma=1$. In addition, by doing $k=0$ in the previous formulas the expressions for the classical hyperbolic functions arise. For example, the Pythagorean Theorem (or fundamental identity) takes the well-known form $[\cosh(x)]^2-[\sinh(x)]^2=1$ and the identities for the half argument are $[\cosh(x)]^2=\frac{\cosh(2x)+1}{2}$, $[\sinh(x)]^2=\frac{\cosh(2x)-1}{2}$.

Therefore these k-Fibonacci hyperbolic functions generalize the classical hyperbolic functions, for a general value $k\in R$.

2.2. The k-Fibonacci hyperbolic functions and the k-Fibonacci numbers

The k-Fibonacci hyperbolic functions are related with the k-Fibonacci numbers, and hence, this relation appears in many expressions. Some of these identities are presented here.

By definition, the k-Fibonacci numbers verify that $F_{k,0}=0$, $F_{k,1}=1$, $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$. An analogous equation for the k-Fibonacci hyperbolic functions is the following:

Proposition 6 (*Recursive relation*).

$$\mathrm{sF}_k\mathrm{h}(x+1)=k\cdot\mathrm{cF}_k\mathrm{h}(x)+\mathrm{sF}_k\mathrm{h}(x-1)$$

Proof. Since $\sigma^2=k\sigma+1$, then $k\sigma^x+\sigma^{x-1}=\sigma^{x-1}(k\sigma+1)=\sigma^{x+1}$. In addition, $k\sigma^{-x}-\sigma^{-x-1}=\frac{k-\sigma}{\sigma^x}=\frac{k\sigma-\sigma^2}{\sigma^{x+1}}=\frac{-1}{\sigma^{x+1}}$.

Therefore $k\cdot\mathrm{cF}_k\mathrm{h}(x)+\mathrm{sF}_k\mathrm{h}(x-1)=\frac{\sigma^{x+1}-\sigma^{-(x+1)}}{\sigma+\sigma^{-1}}=\mathrm{sF}_k\mathrm{h}(x+1)$. □

Catalan's identity for the k-Fibonacci numbers [25] is

$$F_{k,n-r}F_{k,n+r}-F_{k,n}^2=(-1)^{n+1-r}F_{k,r}^2$$

and for the k-Fibonacci hyperbolic functions results:

Proposition 7 (*Catalan's identity*).

$$\mathrm{cF}_k\mathrm{h}(x-r)\cdot\mathrm{cF}_k\mathrm{h}(x+r)-[\mathrm{cF}_k\mathrm{h}(x)]^2=[\mathrm{sF}_k\mathrm{h}(r)]^2$$

Proof. By definition, the left hand side (LHS) is

$$\begin{aligned}(\mathrm{LHS})&=\frac{(\sigma^{x-r}+\sigma^{-x+r})(\sigma^{x+r}+\sigma^{-x-r})-(\sigma^x+\sigma^{-x})^2}{(\sigma+\sigma^{-1})^2}=\frac{\sigma^{2x}+\sigma^{-2r}+\sigma^{2r}+\sigma^{-2x}-\sigma^{2x}-2-\sigma^{-2x}}{(\sigma+\sigma^{-1})^2}\\&=\frac{(\sigma^r-\sigma^{-r})^2}{(\sigma+\sigma^{-1})^2}=[\mathrm{sF}_k\mathrm{h}(r)]^2\qquad\square\end{aligned}$$

In a similar way it can be obtained:

Corollary 8.

$$\begin{aligned}&\mathrm{cF}_k\mathrm{h}(x-r)\cdot\mathrm{cF}_k\mathrm{h}(x+r)-[\mathrm{sF}_k\mathrm{h}(x)]^2=[\mathrm{cF}_k\mathrm{h}(r)]^2\\&\mathrm{sF}_k\mathrm{h}(x-r)\cdot\mathrm{sF}_k\mathrm{h}(x+r)-[\mathrm{sF}_k\mathrm{h}(x)]^2=-[\mathrm{sF}_k\mathrm{h}(r)]^2\\&\mathrm{sF}_k\mathrm{h}(x-r)\cdot\mathrm{sF}_k\mathrm{h}(x+r)-[\mathrm{cF}_k\mathrm{h}(x)]^2=-[\mathrm{cF}_k\mathrm{h}(r)]^2\end{aligned}$$

By doing $r=1$ into classical Catalan's identity it is straightforwardly obtained for the k-Fibonacci numbers Cassini's or Simson's identity [25]:

$$F_{k,n-1}F_{k,n+1}-F_{k,n}^2=(-1)^n$$

The corresponding identity for the k-Fibonacci hyperbolic functions is

Proposition 9 (*Cassini's or Simson's identity*).

$$\begin{aligned}&\mathrm{cF}_k\mathrm{h}(x-1)\cdot\mathrm{cF}_k\mathrm{h}(x+1)-[\mathrm{sF}_k\mathrm{h}(x)]^2=1\\&\mathrm{sF}_k\mathrm{h}(x-1)\cdot\mathrm{sF}_k\mathrm{h}(x+1)-[\mathrm{cF}_k\mathrm{h}(x)]^2=-1\end{aligned}$$

d'Ocagne's identity for the k-Fibonacci numbers [25] is

$$F_{k,m}F_{k,n+1}-F_{k,m+1}F_{k,n}=(-1)^nF_{k,m-n}\quad\text{if } m>n$$

which is generalized for the k-Fibonacci hyperbolic functions as:

Proposition 10 (*d'Ocagne's identity*).

$$\mathrm{cF}_k\mathrm{h}(x)\cdot\mathrm{cF}_k\mathrm{h}(y+r)-\mathrm{sF}_k\mathrm{h}(x+r)\cdot\mathrm{sF}_k\mathrm{h}(y)=\mathrm{cF}_k\mathrm{h}(r)\cdot\mathrm{cF}_k\mathrm{h}(x-y)\tag{5}$$

$$\mathrm{cF}_k\mathrm{h}(x)\cdot\mathrm{sF}_k\mathrm{h}(y+r)-\mathrm{cF}_k\mathrm{h}(x+r)\cdot\mathrm{sF}_k\mathrm{h}(y)=\mathrm{sF}_k\mathrm{h}(r)\cdot\mathrm{cF}_k\mathrm{h}(x-y)\tag{6}$$

Proof. Noting by (LHS) the left hand side of Eq. (5) we have:

$$\begin{aligned}(\mathrm{LHS})&=\frac{(\sigma^x+\sigma^{-x})(\sigma^{y+r}+\sigma^{-y-r})-(\sigma^{x+r}-\sigma^{-x-r})(\sigma^y-\sigma^{-y})}{(\sigma+\sigma^{-1})^2}=\frac{\sigma^{x-y-r}+\sigma^{-x+y+r}+\sigma^{x-y+r}+\sigma^{-x+y-r}}{(\sigma+\sigma^{-1})^2}\\&=\frac{\sigma^{-r}(\sigma^{x-y}+\sigma^{-(x-y)})+\sigma^r(\sigma^{x-y}+\sigma^{-(x-y)})}{(\sigma+\sigma^{-1})^2}=\frac{\sigma^r+\sigma^{-r}}{\sigma+\sigma^{-1}}\cdot\frac{\sigma^{x-y}+\sigma^{-(x-y)}}{\sigma+\sigma^{-1}}=\mathrm{cF}_k\mathrm{h}(r)\cdot\mathrm{cF}_k\mathrm{h}(x-y)\end{aligned}$$

Eq. (6) can be deduced similarly. $\square$

Notice that by doing $r=1$ in Eq. (5) it is obtained:

Corollary 11.

$$\mathrm{cF}_k\mathrm{h}(x)\cdot\mathrm{cF}_k\mathrm{h}(y+1)-\mathrm{sF}_k\mathrm{h}(x+1)\cdot\mathrm{sF}_k\mathrm{h}(y)=\mathrm{cF}_k\mathrm{h}(x-y)$$

3. The quasi-sine *k*-Fibonacci function

Binet's formula for the *k*-Fibonacci sequence establishes that the general term of this sequence can be written as $F_{k,n} = \frac{\sigma^n - (-1)^n \sigma^{-n}}{\sigma + \sigma^{-1}}$, where $\sigma = \sigma_k$ is the positive root of the characteristic equation [25]. On the other hand, we have defined here the *k*-Fibonacci hyperbolic sine function as $\mathrm{sF}_k\mathrm{h}(x) = \frac{\sigma^x - \sigma^{-x}}{\sigma + \sigma^{-1}}$. Therefore, function $\mathrm{sF}_k\mathrm{h}(x)$, if x is an even integer number, takes the value corresponding to the *k*-Fibonacci sequence, that is $\mathrm{sF}_k\mathrm{h}(x) = F_{k,n}$. Therefore, and taking into account that $\cos(n\pi) = (-1)^n$ it is natural to introduce the following definition.

Definition 12 [23]. The quasi-sine *k*-Fibonacci function is defined by

$$\mathrm{FF}_k(x) = \frac{\sigma^x - \cos(\pi x)\sigma^{-x}}{\sigma + \sigma^{-1}}$$

Notice that $\mathrm{FF}_k(n) = F_{k,n}$ for all integer n, so the points of the form $(n, F_{k,n})$ belong to the graphics of these functions. Fig. 2 shows the graphics of $\mathrm{FF}_k(x)$ for $k = 1, 2, 3$.

Note that although the general shape of these functions is quite similar, however they differ in many details as the values of their relative extremes and the number of such extremes. For example, for positive values of x it is observed that $\mathrm{FF}_1(x)$ presents an unique maximum at $x = 1.09458\ldots$ and an unique minimum at $x = 1.67669\ldots$ while for values of $k > 1$ function $\mathrm{FF}_k(x)$ does not have relative extremes for $x > 0$. In addition the number of relative extremes for $x > 0$ decreases when k increases. For example, for $k = 0.8$ there is a unique maximum and a unique minimum; however, for $k = 0.5$ there are two maximums and two minimums; and for $k = 0.25$ there are six maximums and six minimums; and so on. As a matter of example, Fig. 3 shows the graphic of $\mathrm{FF}_k(x)$ for $k = 0.01$.

Many properties of these functions can be investigated, even from an experimental point of view. For instance, for $k = 1$, then $\mathrm{FF}_1(x) = F_n$ for any integer $x = n$, and since $F_1 = F_2 = 1$ and $F_m > F_n$ for $m > n > 2$, function $\mathrm{FF}_1(x)$ presents one maximum and one minimum into the interval $(1, 2)$, and no more extremes for $x > 2$. However, for $k \in N - \{1, 2\}$, function $\mathrm{FF}_k(x)$ increases for $x \geqslant 0$ and, hence, it has not any extreme in $[0, +\infty)$. Therefore, by continuity, there should be some value of k, say k_0, between *1* and *2* such that for any $k > k_0$, function $\mathrm{FF}_k(x)$ has not extremes for positive x. By using MATHEMATICA© this value can be approximately calculated: $k_0 \approx 1.282974\ldots$.

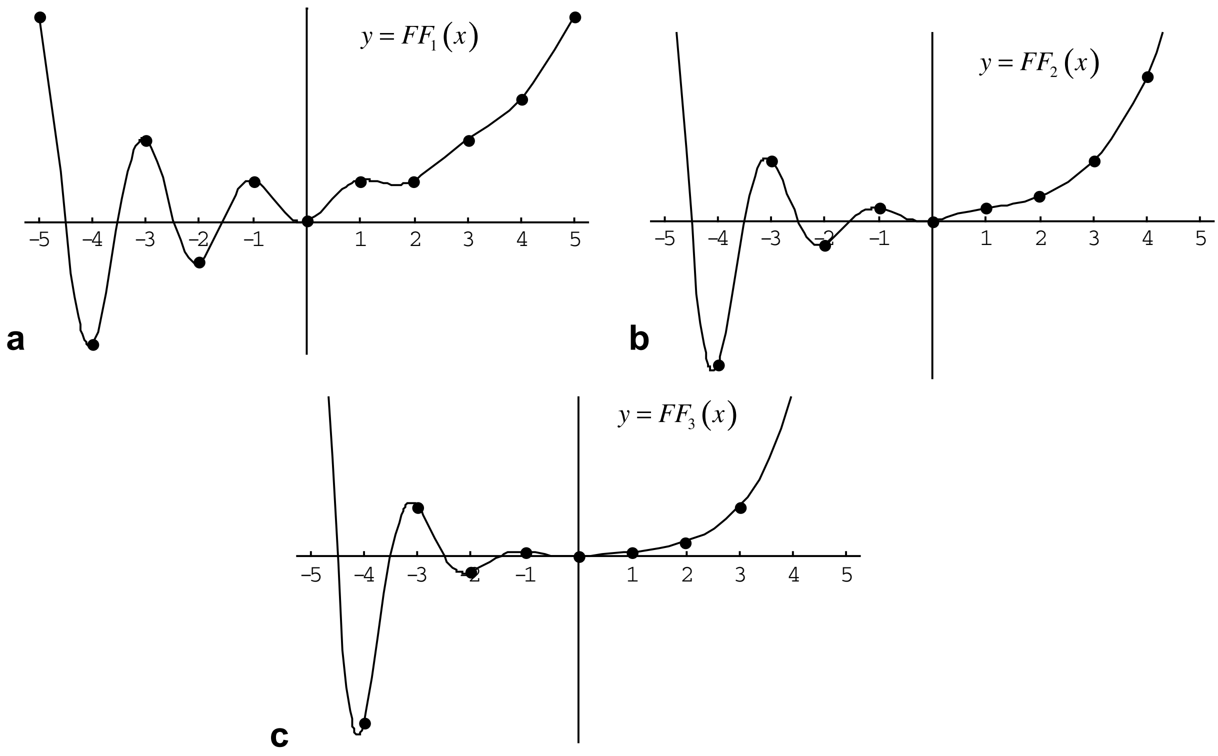

Fig. 2. The quasi-sine *k*-Fibonacci functions, $\mathrm{FF}_k(x)$, for $k = 1, 2, 3$.

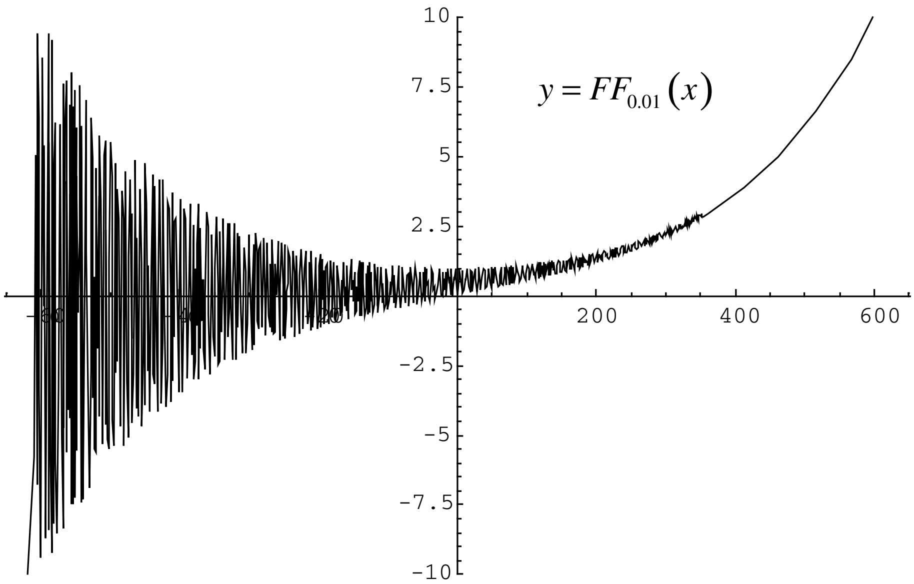

Fig. 3. The quasi-sine 0.01-Fibonacci function.

Finally, the infinite relative extreme points of these functions for $x < 0$ are presented at different values of x. For example, the first minimum for $x < 0$ and $k = 1$ is at $x = -1.0305$ while for $k = 2$ is at $x = -1.07411$, and for $k = 3$ is at $x = -1.10766$.

Fig. 4 shows the graphics of $\mathrm{FF}_k(x)$ for $k = 1, 2$ along with their evolving tangent curves which are precisely the k-Fibonacci cosine and sine hyperbolic functions. Notice that the tangent points are of the form $(n, F_{k,n})$ for $n \in \mathbf{Z}$.

3.1. *The quasi-sine k-Fibonacci functions and the k-Fibonacci numbers*

There are several identities for these quasi-sine k-Fibonacci functions which are versions of analogous identities of the k-Fibonacci numbers [19,21,23,25,26]. Here, some of them are exposed.

The first result looks like the recursive relation defining the k-Fibonacci numbers.

Theorem 13 (*Recursive relation*).

$$\mathrm{FF}_k(x+2) = k \cdot \mathrm{FF}_k(x+1) + \mathrm{FF}_k(x)$$

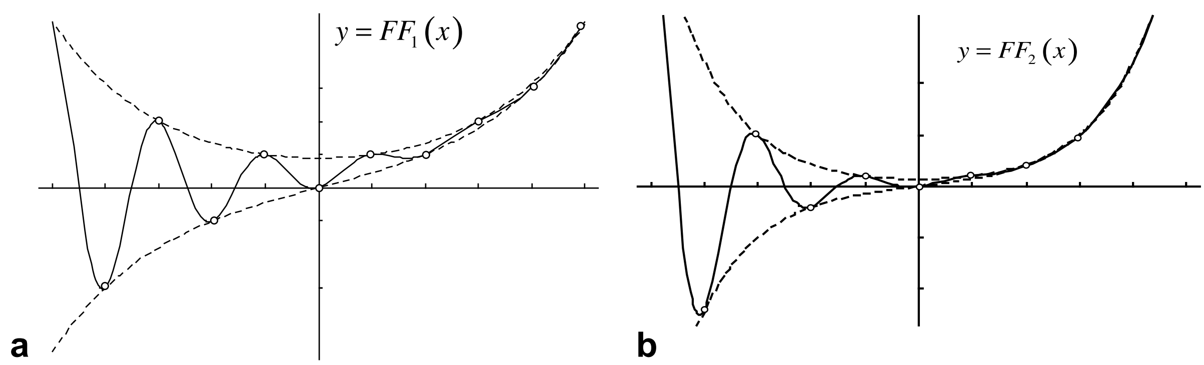

Fig. 4. The quasi-sine k-Fibonacci functions for $k = 1, 2$ with their evolving tangent curves.

Proof.

$$\begin{aligned} k\mathrm{FF}_k(x+1)+\mathrm{FF}_k(x) &= k\frac{\sigma^{x+1}-\cos(\pi(x+1))\sigma^{-x-1}}{\sigma+\sigma^{-1}}+\frac{\sigma^{x}-\cos(\pi x)\sigma^{-x}}{\sigma+\sigma^{-1}} \\ &= k\frac{\sigma^{x+1}+\cos(\pi x)\sigma^{-x-1}}{\sigma+\sigma^{-1}}+\frac{\sigma^{x}-\cos(\pi x)\sigma^{-x}}{\sigma+\sigma^{-1}}=\frac{\sigma^{x}(k\sigma+1)+\cos(\pi x)\sigma^{-x-2}(k\sigma-\sigma^2)}{\sigma+\sigma^{-1}} \\ &= \frac{\sigma^{x+2}+\cos(\pi x+2\pi)\sigma^{-x-2}(k\sigma-k\sigma-1)}{\sigma+\sigma^{-1}}=\mathrm{FF}_k(x+2) \qquad \square \end{aligned}$$

Similarly to Catalan's identity for the k-Fibonacci numbers, for these functions we obtain:

Theorem 14 (Catalan's identity). *If* $r\in\mathbf{Z}$, *then*:

$$FF_k(x+r)\cdot FF_k(x-r)-[FF_k(x)]^2=(-1)^{r+1}\cos(\pi x)[FF_k(r)]^2$$

Proof.

$$\begin{aligned} \mathrm{FF}_k(x+r)\cdot\mathrm{FF}_k(x-r)-[\mathrm{FF}_k(x)]^2 &= \frac{\sigma^{x+r}-\cos(\pi(x+r))\sigma^{-x-r}}{\sigma+\sigma^{-1}}\cdot\frac{\sigma^{x-r}-\cos(\pi(x-r))\sigma^{-x+r}}{\sigma+\sigma^{-1}}-\left(\frac{\sigma^{x}-\cos(\pi x)\sigma^{-x}}{\sigma+\sigma^{-1}}\right)^2 \\ &= \frac{\sigma^{x+r}+(-1)^{r+1}\cos(\pi x)\sigma^{-x-r}}{\sigma+\sigma^{-1}}\cdot\frac{\sigma^{x-r}+(-1)^{r+1}\cos(\pi x)\sigma^{-x+r}}{\sigma+\sigma^{-1}} \\ &\quad -\frac{\sigma^{2x}-2\cos(\pi x)+\cos^2(\pi x)\sigma^{-2x}}{(\sigma+\sigma^{-1})^2} \\ &= \frac{(-1)^{r+1}\cos(\pi x)\sigma^{2r}+(-1)^{r+1}\cos(\pi x)\sigma^{-2r}+2\cos(\pi x)}{(\sigma+\sigma^{-1})^2} \\ &= \frac{(-1)^{r+1}\cos(\pi x)(\sigma^{r}-\cos(\pi r)\sigma^{-r})^2}{(\sigma+\sigma^{-1})^2}=(-1)^{r+1}\cos(\pi x)[\mathrm{FF}_k(r)]^2 \qquad \square \end{aligned}$$

Note that if in the previous result x is an integer, n, then Catalan's identity for the k-Fibonacci numbers appears. Also if $r=1$ results:

$$\mathrm{FF}_k(x+1)\cdot\mathrm{FF}_k(x-1)-[\mathrm{FF}_k(x)]^2=\cos(\pi x)$$

which for $x=n$ an integer gives us Cassini's or Simson's identity [25] for the k-Fibonacci numbers:

$$F_{k,n+1}\cdot F_{k,n-1}-F_{k,n}^2=(-1)^n$$

Theorem 15 (Asymptotic quotient). *For any integer* r, *it is*

$$\lim_{x\to\infty}\frac{FF_k(x+r)}{FF_k(x)}=\sigma^r$$

Proof. Taking into account that $\sigma>1$, then:

$$\lim_{x\to\infty}\frac{\mathrm{FF}_k(x+r)}{\mathrm{FF}_k(x)}=\lim_{x\to\infty}\frac{\sigma^{x+r}-\cos(\pi(x+r))\sigma^{-x-r}}{\sigma^{x}-\cos(\pi x)\sigma^{-x}}=\lim_{x\to\infty}\frac{\sigma^{r}+(-1)^{r+1}\cos(\pi x)\frac{1}{\sigma^{2x+r}}}{1-\cos(\pi x)\frac{1}{\sigma^{2x}}}=\sigma^r \qquad \square$$

In particular, if $k=1$, then $\sigma=\phi$ and $\lim_{x\to\infty}\frac{\mathrm{FF}_k(x+1)}{\mathrm{FF}_k(x)}=\phi$.

4. 3D curves and surfaces for the *k*-Fibonacci hyperbolic functions

We shall introduce here several curves and surfaces naturally related with the k-Fibonacci functions studied before.

Definition 16. The following complex valued function is called the three-dimensional k-Fibonacci spiral:

$$\mathrm{CFF}_k(x) = \frac{\sigma^x - \cos(\pi x)\sigma^{-x}}{\sigma + \sigma^{-1}} + \mathrm{i}\frac{\sin(\pi x)\sigma^{-x}}{\sigma + \sigma^{-1}}$$

where σ is the positive root of the characteristic equation of the k-Fibonacci sequence: $r^2 - kr - 1 = 0$.

Note that $\Re(\mathrm{CFF}_k(x)) = \mathrm{FF}_k(x)$, which is the quasi-sine k-Fibonacci function. The shape of the three-dimensional k-Fibonacci spiral reminds a three-dimensional spiral as shown in Fig. 5.

Function $\mathrm{CFF}_k(x)$ can be written also as $\mathrm{CFF}_k(x) = \frac{\sigma^x + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x\right)}\sigma^{-x}}{\sigma+\sigma^{-1}}$.
Some of the properties of the function $\mathrm{CFF}_k(x)$ are the following:

- If $k = 1$, the three-dimensional Fibonacci spiral introduced by Stakhov [21] is obtained. Note that, in this case, $\mathrm{CFF}_k(x)$ can be written also as $\mathrm{CFF}_1(x) = \mathrm{CFF}(x) = \frac{\phi^x + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x\right)}\phi^{-x}}{\sqrt{5}}$, where ϕ is the golden ratio.
- If x is an integer, the imaginary part of the three-dimensional k-Fibonacci spiral is zero, and hence $\mathrm{CFF}_k(n)$ results in Binet's formula for the general term of the k-Fibonacci sequence [25].
- Finally, as an immediate consequence of the last fact, for every k it is $\mathrm{CFF}_k(0) = 0$ and $\mathrm{CFF}_k(1) = 1$.

By the definition of these functions we will prove the following theorem.

Theorem 17 (Recurrence relation). *The three-dimensional k-Fibonacci spirals verify the following Fibonacci-type recurrence relation*:

$$\mathrm{CFF}_k(x+2) = k \cdot \mathrm{CFF}_k(x+1) + \mathrm{CFF}_k(x).$$

Proof. Let us note by (RHS) the right hand side of the identity to prove. Taking into account that $\sigma > 1$, and since $k\sigma + 1 = \sigma^2$, so $\sigma - k = \frac{1}{\sigma}$, we have:

$$\begin{aligned}(\mathrm{RHS}) &= k \cdot \frac{\sigma^{x+1} + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x-1\right)}\sigma^{-x-1}}{\sigma + \sigma^{-1}} + \frac{\sigma^x + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x\right)}\sigma^{-x}}{\sigma + \sigma^{-1}} = \frac{\sigma^x(k\sigma+1) + \mathrm{i}e^{-\mathrm{i}\pi x}\sigma^{-x-1}(ke^{-i\frac{\pi}{2}} + \sigma e^{i\frac{\pi}{2}})}{\sigma + \sigma^{-1}} \\ &= \frac{\sigma^{x+2} + \mathrm{i}e^{-\mathrm{i}\pi x}\sigma^{-x-1}(-k+\sigma)i}{\sigma + \sigma^{-1}} = \frac{\sigma^{x+2} + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-(x+2)\right)}\sigma^{-x-2}}{\sigma + \sigma^{-1}} = \mathrm{CFF}_k(x+2) \qquad \square\end{aligned}$$

Since we have shown Catalan's identity for the k-Fibonacci hyperbolic functions, and also for the quasi-sine Fibonacci functions, an analogous identity may be demonstrated for the three-dimensional k-Fibonacci spiral.

Theorem 18 (Catalan's identity). *If r is an integer, then the three-dimensional k-Fibonacci spiral verifies*:

$$\mathrm{CFF}_k(x-r) \cdot \mathrm{CFF}_k(x+r) - [\mathrm{CFF}_k(x)]^2 = (-1)^{r+1}[\mathrm{CFF}_k(r)]^2.$$

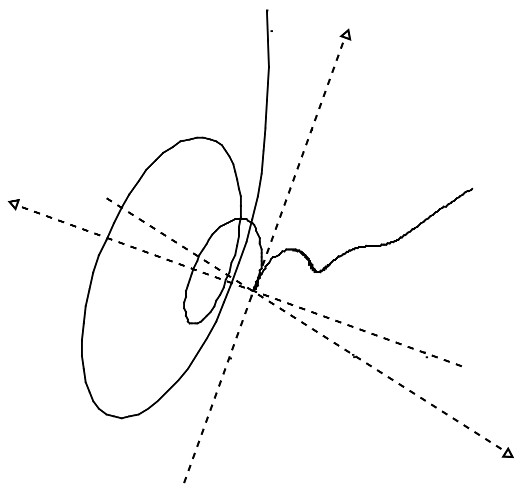

Fig. 5. The three-dimensional k-Fibonacci spiral, for $k = 1$.

Proof. Let us note by (LHS) the left hand side of the identity to prove.

$$\text{(LHS)} = \frac{\sigma^{x-r} + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x+r\right)}\sigma^{-x+r}}{\sigma+\sigma^{-1}} \cdot \frac{\sigma^{x+r} + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x-r\right)}\sigma^{-x-r}}{\sigma+\sigma^{-1}} - \left(\frac{\sigma^{x} + \mathrm{i}e^{\mathrm{i}\pi\left(\frac{1}{2}-x\right)}\sigma^{-x}}{\sigma+\sigma^{-1}}\right)^2$$

But $e^{\mathrm{i}\pi\left(\frac{1}{2}-x\right)} = \mathrm{i}e^{-\mathrm{i}\pi x}$, and $e^{\mathrm{i}\pi r} = (-1)^r$, $e^{\mathrm{i}\left(\frac{\pi}{2}\pm\pi r\right)} = \mathrm{i}(-1)^r$ since r is an integer. Therefore,

$$\begin{aligned}\text{(LHS)} &= \frac{(\sigma^{x-r} + (-1)^{r+1}e^{-\mathrm{i}\pi x}\sigma^{-x+r})(\sigma^{x+r} + (-1)^{r+1}e^{-\mathrm{i}\pi x}\sigma^{-x-r}) - (\sigma^{2x} - 2\cdot e^{-\mathrm{i}\pi x} + e^{-\mathrm{i}2\pi x}\sigma^{-2x})}{(\sigma+\sigma^{-1})^2} \\ &= \frac{\sigma^{2x} + (-1)^{r+1}e^{-\mathrm{i}\pi x}\sigma^{-2r} + (-1)^{r+1}e^{-\mathrm{i}\pi x}\sigma^{2r} + e^{-\mathrm{i}2\pi x}\sigma^{-2x} - \sigma^{2x} + 2e^{-\mathrm{i}\pi x} - e^{-\mathrm{i}2\pi x}\sigma^{-2x}}{(\sigma+\sigma^{-1})^2} \\ &= (-1)^{r+1}\frac{e^{-\mathrm{i}\pi x}(\sigma^{2r} - (-1)^r 2 + \sigma^{-2r})}{(\sigma+\sigma^{-1})^2} = (-1)^{r+1}[\mathrm{CFF}_k(r)]^2 \qquad \square\end{aligned}$$

where if $r = 1$ Simson's identity results:

$$\mathrm{CFF}_k(x-1)\cdot\mathrm{CFF}_k(x+1) - [\mathrm{CFF}_k(x)]^2 = (-1)^{r+1}$$

Using a similar process, it can be proven the following theorem.

Theorem 19 (d'Ocagne's identity). *If r is an integer, then the three-dimensional k-Fibonacci spiral verifies*:

$$\mathrm{CFF}_k(x+r)\cdot\mathrm{CFF}_k(x+1) - \mathrm{CFF}_k(x+r+1)\cdot\mathrm{CFF}_k(x) = e^{-\mathrm{i}\pi x}\mathrm{CFF}_k(r).$$

If $x = r = n$ previous formula reduces to

$$\mathrm{CFF}_k(2n)\cdot\mathrm{CFF}_k(n+1) - \mathrm{CFF}_k(2n+1)\cdot\mathrm{CFF}_k(n) = (-1)^n\mathrm{CFF}_k(n)$$

or equivalently

$$F_{k,2n}\cdot F_{k,n+1} - F_{k,2n+1}\cdot F_{k,n} = (-1)^n F_{k,n}$$

4.1. The Metallic Shofars

Real and imaginary parts of the three-dimensional k-Fibonacci spiral are, respectively $\Re(\mathrm{CFF}_k(x)) = \frac{\sigma^x - \cos(\pi x)\sigma^{-x}}{\sigma+\sigma^{-1}} = y(x)$, $\Im(\mathrm{CFF}_k(x)) = \frac{\sin(\pi x)\sigma^{-x}}{\sigma+\sigma^{-1}} = z(x)$. By considering Y axis as real axis, and Z axis as imaginary axis we can write the following system of equations:

$$\left\{\begin{aligned} y(x) - \tfrac{\sigma^x}{\sigma+\sigma^{-1}} &= -\tfrac{\cos\pi x\sigma^{-x}}{\sigma+\sigma^{-1}} \\ z(x) &= \tfrac{\sin\pi x\sigma^{-x}}{\sigma+\sigma^{-1}} \end{aligned}\right\}$$

By summing up the squares of previous expressions and considering y and z as independent variables the following equation is obtained:

$$\left(y - \frac{\sigma^x}{\sigma+\sigma^{-1}}\right)^2 + z^2 = \left(\frac{\sigma^{-x}}{\sigma+\sigma^{-1}}\right)^2 \tag{7}$$

Eq. (7) corresponds to a surface which have been called in the case $k = 1$ the *Golden Shofar* [21] with equation $\left(y - \frac{\phi^x}{\sqrt{5}}\right)^2 + z^2 = \frac{1}{5\phi^{2x}}$, where ϕ is the golden ratio. For the case $k = 2$ the *Silver Shofar* with equation $\left(y - \frac{\varphi^x}{\sqrt{8}}\right)^2 + z^2 = \frac{1}{8\varphi^{2x}}$ is obtained. Finally, for $k = 3$ results the *Bronze Shofar* with equation $\left(y - \frac{\psi^x}{\sqrt{13}}\right)^2 + z^2 = \frac{1}{13\psi^{2x}}$.

Fig. 6 shows the Metallic Shofars for $k = 1, 2, 3$, while Fig. 7 shows the Golden Shofar and its projections on the coordinate planes.

Remark 20. Let z as in Eq. (7), then $z^2 = (cF_kh(x) - y)(y - sF_kh(x))$.

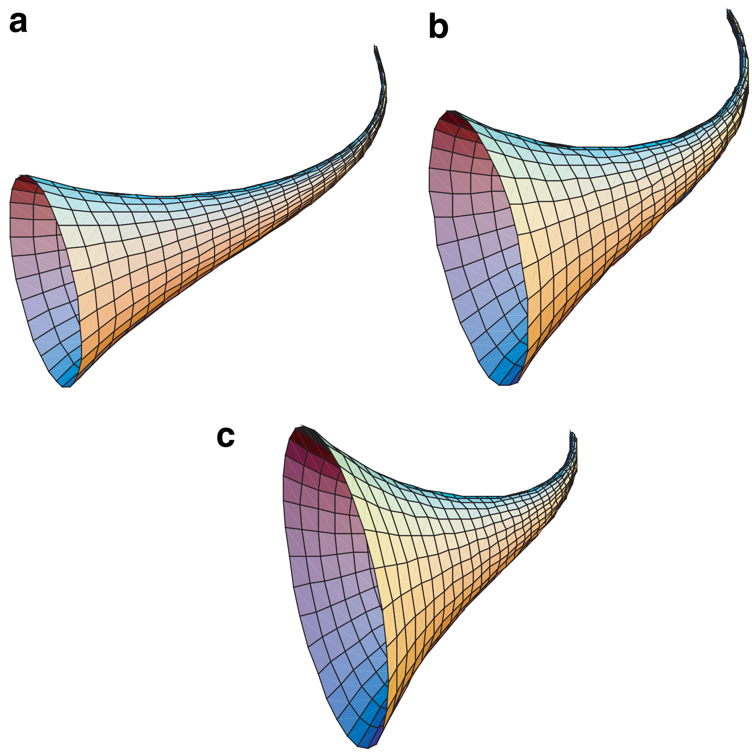

Fig. 6. The Metallic Shofars: (a) Golden Shofar ($k = 1$); (b) Silver Shofar ($k = 2$); (c) Bronze Shofar ($k = 3$).

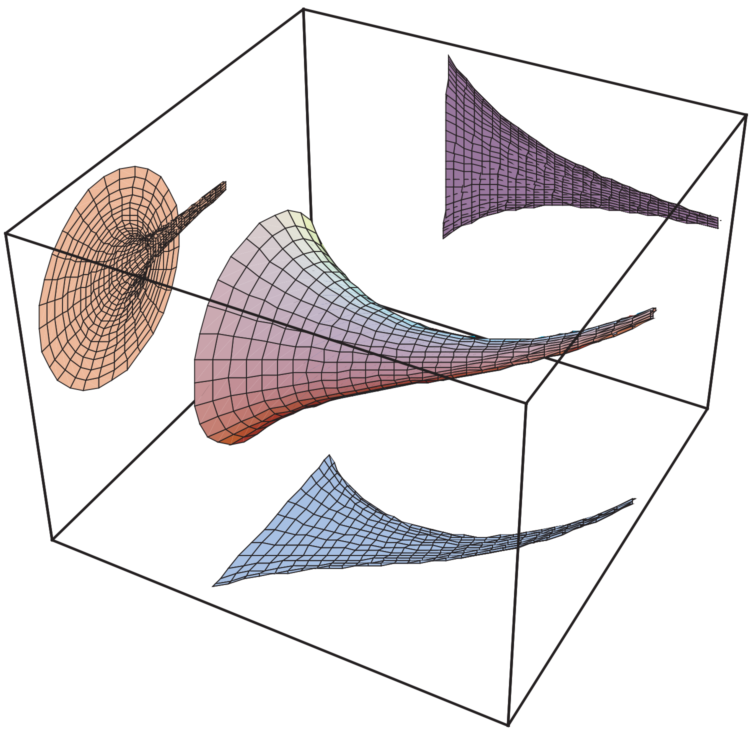

Fig. 7. The Golden Shofar and its projections on the coordinate planes.

5. Conclusions

New k-Fibonacci hyperbolic functions have been introduced and studied here. These new functions are naturally related with the k-Fibonacci sequences. Several properties of these functions have been deduced and related with the analogous identities for the k-Fibonacci numbers. In addition, the k-Fibonacci hyperbolic functions are used to introduce the so-called Metallic Shofars which extend the Golden Shofar.

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