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On k-Fibonacci sequences and polynomials and their derivatives

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Abstract

The k-Fibonacci polynomials are the natural extension of the k-Fibonacci numbers and many of their properties admit a straightforward proof. Here in particular, we present the derivatives of these polynomials in the form of convolution of k-Fibonacci polynomials. This fact allows us to present in an easy form a family of integer sequences in a new and direct way. Many relations for the derivatives of Fibonacci polynomials are proven.

1. Introduction

There is a huge interest of modern science in the application of the Golden Section and Fibonacci numbers [1–10]. The Fibonacci numbers F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \ldots\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. On the other hand the ratio of two consecutive Fibonacci numbers converges to the Golden Mean, or Golden Section, $\phi = \frac{1+\sqrt{5}}{2}$, which appears in modern research, particularly physics of the high energy particles [11,12] or theoretical physics [13–19].

The paper presented here was initially originated for the astonishing presence of the Golden Section in a recursive partition of triangles in the context of the finite element method and triangular refinements. In [20] we showed the relation between the 4-triangle longest-edge partition and the k-Fibonacci numbers, as another example of the relation between geometry and numbers. On the other hand in [21] the k-Fibonacci numbers were given in an explicit way and, by easy arguments, many properties were proven. In particular the k-Fibonacci numbers were related with the so-called Pascal 2-triangle.

From this point on, the present paper is organized as follows. In Section 2 a brief summary of the previous results obtained in References [20–22] is given. Section 3 is focused on the k-Fibonacci polynomials which are the natural extension of the k-Fibonacci numbers and many of their properties admit a straightforward proof. Here, in particular we relate the Catalan's numbers with the coefficients of the successive powers x^n written as linear combination of Fibonacci polynomials. Section 4 presents many formulas and relations for the derivatives of the Fibonacci polynomials. For example, their derivatives are given as convolution of Fibonacci polynomials. This fact allows us to present a family of integer sequences in a new and direct way.

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2. The *k*-Fibonacci numbers and properties

The k-Fibonacci numbers have been defined in [21] for any real number k as follows.

Definition 1. For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for } n \geqslant 1. \tag{1}$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1 \tag{2}$$

Note that if k is a real variable x then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \left\{ \begin{array}{cl} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ xF_n(x) + F_{n-1}(x) & \text{if } n > 1 \end{array} \right\}$$

Particular cases of the k-Fibonacci sequence are

- If $k = 1$, the classical Fibonacci sequence is obtained:

  $F_0 = 0, F_1 = 1,$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 1$:
  $\{F_n\}_{n\in\mathbf{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$
- If $k = 2$, the Pell sequence appears:

  $P_0 = 0, P_1 = 1,$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geqslant 1$:
  $\{P_n\}_{n\in\mathbf{N}} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}$
- If $k = 3$, the following sequence appears:

  $H_0 = 0, H_1 = 1,$ and $H_{n+1} = 3H_n + H_{n-1}$ for $n \geqslant 1$:
  $\{H_n\}_{n\in\mathbf{N}} = \{0, 1, 3, 10, 33, 109, \ldots\}$

The well-known Binet's formula in the *Fibonacci numbers theory* [1,8,21] allows us to express the k-Fibonacci number in function of the roots r_1 and r_2 of the characteristic equation, associated to the recurrence relation (1) $r^2 = kr + 1$:

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{3}$$

If σ denotes the positive root of the characteristic equation, the general term may be written in the form $F_{k,n} = \frac{\sigma^n - \sigma^{-n}}{\sigma + \sigma^{-1}}$, and the limit of the quotient of two terms is

$$\lim_{n\to\infty} \frac{F_{k,n+r}}{F_{k,n}} = \sigma^r \tag{4}$$

In addition, the general term of the k-Fibonacci sequence may be obtained by the formula:

$$F_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-2i-1} (k^2 + 4)^i \tag{5}$$

or, equivalently, by

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i} \tag{6}$$

Table 1
The first k-Fibonacci numbers

$F_{k,1} = 1$
$F_{k,2} = k$
$F_{k,3} = k^2 + 1$
$F_{k,4} = k^3 + 2k$
$F_{k,5} = k^4 + 3k^2 + 1$
$F_{k,6} = k^5 + 4k^3 + 3k$
$F_{k,7} = k^6 + 5k^4 + 6k^2 + 1$
$F_{k,8} = k^7 + 6k^5 + 10k^3 + 4k$

2.1. The k-Fibonacci numbers and the Pascal 2-triangle

From the definition of the k-Fibonacci numbers, the first of them are presented in Table 1. From these expressions, one may deduce the value of any k-Fibonacci number by simple substitution on the corresponding $F_{k,n}$. For example, the seventh element of the 4-Fibonacci sequence, $\{F_{4,n}\}_{n\in N}$, is $F_{4,7} = 4^6 + 5 \cdot 4^4 + 6 \cdot 4^2 + 1 = 5473$.

By doing $k = 1, 2, 3, \ldots$ the respective k-Fibonacci sequences are obtained whose first elements are

$$\{F_{1,n}\}_{n\in N} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 \ldots\}$$
$$\{F_{2,n}\}_{n\in N} = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378 \ldots\}$$
$$\{F_{3,n}\}_{n\in N} = \{0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, \ldots\}$$
$$\{F_{4,n}\}_{n\in N} = \{0, 1, 4, 17, 72, 305, 1292, 5473, 23184, 98209, 416020, \ldots\}$$

Sequence $\{F_{1,n}\}$ is the classical Fibonacci sequence and $\{F_{2,n}\}$ is the Pell sequence. It is worthy to be noted that only the first 11 k-Fibonacci sequences are referenced in *The On-Line Encyclopedia of Integer Sequences* [23] with the numbers given in Table 2. For k even with $12 \leqslant k \leqslant 62$ sequences $\{F_{k,n}\}$ are referenced without the first term $F_{k,0} = 0$ in [23].

It is worthy to be noted that the coefficients arising in the previous list, see Table 1, can be written in triangular position, in such a way that every side of the triangle is double, and for this reason this triangle has been called *Pascal* 2-*triangle* [21]. See Table 3.

Table 2
The first 11 k-Fibonacci sequences as numbered in *The On-Line Encyclopedia of Integer Sequences* [23]

$\{F_{1,n}\}$	*A*000045	$\{F_{7,n}\}$	*A*054413
$\{F_{2,n}\}$	*A*000129	$\{F_{8,n}\}$	*A*041025
$\{F_{3,n}\}$	*A*006190	$\{F_{9,n}\}$	*A*099371
$\{F_{4,n}\}$	*A*001076	$\{F_{10,n}\}$	*A*041041
$\{F_{5,n}\}$	*A*052918	$\{F_{11,n}\}$	*A*049666
$\{F_{6,n}\}$	*A*005668		

Table 3
The Pascal 2-triangle

1.							1						
2.							1						
3.						1		1					
4.						1		2					
5.					1		3		1				
6.					1		4		3				
7.				1		5		6		1			
8.				1		6		10		4			
9.			1		7		15		10		1		
10.			1		8		21		20		5		
11.		1		9		28		35		15		1	
12.		1		10		36		56		35		6	
13.	1		11		45		84		70		21		1
14.	1		12		55		120		126		56		7

Table 4
New table from the Classical Pascal triangle

1	1	1	1	1	1	1	...
1	2	3	4	5	6	7	...
1	3	6	10	15	21	28	...
1	4	10	20	35	56	84	...
1	5	15	35	70	126	210	...
1	6	21	56	126	252	462	...
				...			

Note that the numbers belonging to the same row of the Pascal 2-triangle are the coefficients of $F_{k,n}$ as they are expressed in Eq. (6). Also the elements belonging to the odd rows in inverse way build the *Modified Numerical Triangle (MNT)* introduced by Trzaska in [24–26], and defined by the recurrence equation $T_{n+2}(x) = (2+x)T_{n+1}(x) - T_n(x)$, for $n = 0, 1, 2, \ldots$, with initial values $T_0(x) = 1$ and $T_1(x) = 1 + x$. On the other hand, the even rows, written in inverse way, form the so-called ($MNT2$) triangle introduced by Trzaska in [26], and defined by the recurrence equation $P_{n+2}(x) = (2+x)P_{n+1}(x) - P_n(x)$, for $n = 0, 1, 2, \ldots$, with initial values $P_0(x) = 1$ and $P_1(x) = 1$.

A simple explanation of the Pascal 2-triangle may be given by considering two sets of points in the coordinate axes $X = \{x = (x,0)/x \in N\}$ and $Y = \{y = (0,y)/y \in N\}$. A path between an x-point and a y-point is the not reversing path in the first quadrant from x to y by horizontal and vertical unit segments. For example, from point $x = (2,0)$ to point $y = (0,1)$ there are three paths: $\{(2,0),(1,0),(0,0),(0,1)\}$, $\{(2,0),(1,0),(1,1),(0,1)\}$, and $\{(2,0),(2,1),(1,1),(0,1)\}$. Then, as it can be easily checked, the diagonals in the Pascal 2-triangle give the number of such paths between an x-point and an y-point.

Each one of the different lines of couples of adjacent numbers on the Pascal 2-triangle as shown in Table 3, from right to left and from first row to bottom is called *double diagonal* [21]. For example, the third double diagonal is $\{1-3, 6-10, 15-21, 28-36, \ldots\}$. These lines are also called *antidiagonals*. In addition, each line of numbers from left to right and from top to bottom is called *simple diagonal*. For example, the third simple diagonal is

$$\{1, 3, 6, 10, 15, 21, 28, 36, \ldots\}.$$

Note that the ith double diagonal is equal to the same order simple diagonal, and, therefore, they can be named diagonal (simple) or antidiagonal (double).

For many properties of the Pascal 2-triangle see, for example [21].

Writing down the diagonals of the classical Pascal 2-triangle in rows is obtained Table 4.

Between the properties of this table, we emphasize the following ones. Each entrance of the row beginning with $\{1, i, \ldots\}$ gives precisely the number of terms in the expansion of $(a_1 + a_2 + a_3 + \cdots + a_i)^n$, for $n = 0, 1, 2, 3, \ldots$, and this number is $\binom{n+i-1}{n}$. Each entrance in the diagonal beginning with $\{1, j, \ldots\}$ reports the number of terms in the expansion of $(a_1 + a_2 + a_3 + \cdots + a_{n+3-j})^n$, for $n \geqslant j-2$, and this number is $\binom{2n+j-4}{n-2}$. For instance, diagonal $\{1, 4, 15, 56, 210, \ldots\}$ means the number of terms in the expansion of $(a_1 + a_2 + a_3 + \cdots + a_{n-1})^n$ for $n \geqslant 2$, which is $\binom{2n-2}{n-2}$. In particular, by dividing each term of this diagonal respectively by $1, 2, 3, 4, \ldots$ the sequence of Catalan's numbers is obtained.

The antidiagonals in Table 4 correspond to the rows of the classical Pascal triangle, which are $\binom{n}{i}$. Term $a_{i,j}$ of Table 4 verifies $a_{i,j} = a_{i-1,j} + a_{i,j-1}$. The square matrices obtained from term $a_{1,1}$ have determinant equal to 1. They are also symmetric with respect to the diagonal $\{1, 2, 6, 20, 70, \ldots\}$.

3. The Fibonacci polynomials

Note that if k is a real variable x then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \left\{ \begin{array}{ll} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ xF_n(x) + F_{n-1}(x) & \text{if } n \geqslant 2 \end{array} \right\} \tag{7}$$

from where the first Fibonacci polynomials are

$$F_1(x) = 1$$
$$F_2(x) = x$$
$$F_3(x) = x^2 + 1$$
$$F_4(x) = x^3 + 2x$$
$$F_5(x) = x^4 + 3x^2 + 1$$
$$F_6(x) = x^5 + 4x^3 + 3x$$
$$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$$
$$F_8(x) = x^7 + 6x^5 + 10x^3 + 4x$$

and from these expressions, as for the k-Fibonacci numbers we can write:

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i} \quad \text{for } n \geqslant 0 \tag{8}$$

Note that $F_{2n}(0) = 0$ and $x = 0$ is the only real root, while $F_{2n+1}(0) = 1$ with no real roots. Also for $x = k \in N$ we obtain the elements of the k-Fibonacci sequences.

By iterating recurrence relation of Formula (7) the following property is straightforwardly deduced.

Proposition 2. *For* $1 \leqslant r \leqslant n-1$ *holds*:

$$F_{n+1}(x) = F_r(x)F_{n-(r-2)}(x) + F_{r-1}(x)F_{n-(r-1)}(x) \tag{9}$$

Proposition 3 (Binet's formula). *The nth Fibonacci polynomial may be written as*

$$F_n(x) = \frac{\sigma^n - (-\sigma)^{-n}}{\sigma + \sigma^{-1}} \tag{10}$$

being $\sigma = \frac{x+\sqrt{x^2+4}}{2}$.

Proof. Note that the characteristic equation for the k-Fibonacci polynomials is $r^2 - x \cdot r - 1 = 0$ with roots $r_1 = \sigma = \frac{x+\sqrt{x^2+4}}{2}$, and $r_2 = -\sigma^{-1}$, from where Formula (10) is deduced. □

From Binet's formula, and in the same way that for the k-Fibonacci numbers [21] the following propositions may be proved:

Proposition 4 (*Asymptotic behaviour of the quotient of consecutive terms*).

$$\text{If } \sigma = \frac{x + \sqrt{x^2+4}}{2}, \text{ then } \lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \sigma \qquad \square$$

As a consequence, the quotient between two consecutive terms of the k-Fibonacci sequence $\{F_{k,n}\} = \{0, 1, k, k^2+1, k^3+2k, \ldots\}$ tends to the positive characteristic root $\sigma = \sigma_k$. For each integer k, $\sigma = \sigma_k$ is called the kth metallic ratio [10]: Golden Ratio, for $k = 1$, Silver Ratio, for $k = 2$, and Bronze Ratio for $k = 3$.

Proposition 5 (Honsberger's formula). *For* n, m *integers it holds*:

$$F_{m+n}(x) = F_{m+1}(x)F_n(x) + F_m(x)F_{n-1}(x) \tag{11}$$

Proof. Eq. (11) follows by changing in Eq. (9) $n - r + 2$ by n and r by $m + 1$. □

In particular,

- For $m = n - 1$ an expression for the polynomial of even degree is obtained: $F_{2n-1}(x) = F_n^2(x) + F_{n-1}^2(x)$.
- For $m = n$ it is obtained:

$$F_{2n}(x) = F_{n+1}(x)F_n(x) + F_n(x)F_{n-1}(x) = F_n(x)(F_{n+1}(x) + F_{n-1}(x))$$

or, equivalently: $F_{2n}(x) = \frac{F_{n+1}^2(x) - F_{n-1}^2(x)}{x}$.

Previous argument may be applied for $m = 2n, 3n, \ldots$ from which it is deduced that the $r \cdot n$ order Fibonacci polynomial is multiple of the n order polynomial, and hence

- $GCD[F_m(x), F_n(x)] = F_{GCD[m,n]}(x)$

Proposition 6 (Catalan's identity). *For n, r integers and n > r, then*

$$F_{n-r}(x)F_{n+r}(x) - F_n^2(x) = (-1)^{n-r-1}F_r^2(x)$$

Proof. By applying Binet's formula (10) to the left-hand side (*LHS*) results:

$$\begin{aligned}(\mathrm{LHS}) &= \frac{\sigma^{n-r} - (-1)^{n-r}\sigma^{-n+r}}{\sigma + \sigma^{-1}} \cdot \frac{\sigma^{n+r} - (-1)^{n+r}\sigma^{-n-r}}{\sigma + \sigma^{-1}} - \left(\frac{\sigma^n - (-1)^n\sigma^{-n}}{\sigma + \sigma^{-1}}\right)^2 \\ &= \frac{\sigma^{2n} - (-1)^{n+r}\sigma^{-2r} - (-1)^{n-r}\sigma^{2r} + \sigma^{-2n}}{(\sigma + \sigma^{-1})^2} - \frac{\sigma^{2n} - 2(-1)^n + \sigma^{-2n}}{(\sigma + \sigma^{-1})^2} \\ &= \frac{(-1)^{n-r-1}\sigma^{-2r} + (-1)^{n-r-1}\sigma^{2r} - 2(-1)^{n-1}}{(\sigma + \sigma^{-1})^2} = (-1)^{n-r-1}\left(\frac{\sigma^r - \sigma^{-r}}{\sigma + \sigma^{-1}}\right)^2 = (\mathrm{RHS}) \qquad \square\end{aligned}$$

Straightforward corollaries of Catalan's identity are

- Cassini's or Simson's identity (by doing $r = 1$): $F_{n-1}(x)F_{n+1}(x) - F_n^2(x) = (-1)^n$.
- By changing n by $4n$ and r by $2n$, results: $F_{2n}(x)(F_{2n}(x) + F_{6n}(x)) = F_{4n}^2(x)$, and therefore the (LHS) is a perfect square [27].
- By changing n by $2n + r$, results: $F_{2n}(x)F_{2n+2r}(x) + F_r^2(x) = F_{2n+r}^2(x)$, and therefore the (LHS) is a perfect square. If $x = 1$ we have $F_n(x) = F_n$ and hence $F_1(1) = F_2(1) = 1$, and the set $\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\}$ is a Diophantine quadruple [28,29] which means that the product of two of them plus +1 is a perfect square. For example: $F_{2n} \cdot F_{2n+2} + 1 = F_{2n+1}^2$, $F_{2n} \cdot F_{2n+4} + 1 = F_{2n+2}^2$, $F_{2n} \cdot 4F_{2n+1}F_{2n+2}F_{2n+3} + 1 = (2F_{2n+1}F_{2n+2} - 1)^2$, $F_{2n+2} \cdot F_{2n+4} + 1 = F_{2n+3}^2$, $F_{2n+2} \cdot 4F_{2n+1}F_{2n+2}F_{2n+3} + 1 = (2F_{2n+2}^2 + 1)^2$, and $F_{2n+4} \cdot 4F_{2n+1}F_{2n+2}F_{2n+3} + 1 = (2F_{2n+2}F_{2n+3} + 1)^2$.

Proposition 7 (General bilinear formula). *For a, b, c, d and r integers, with a + b = c + d*:

$$F_a(x)F_b(x) - F_c(x)F_d(x) = (-1)^r(F_{a-r}(x)F_{b-r}(x) - F_{c-r}(x)F_{d-r}(x)) \tag{12}$$

Proof. As it is well-known if Q is a square matrix and a, b, c and d are real numbers with $a + b = c + d$ then $Q^{a+b} = Q^{c+d}$. Let Q be the square matrix $(R^{k-1} \cdot L)^n$ which was introduced in [20]. Matrix Q for Fibonacci polynomials is of the form $(R^{k-1} \cdot L)^n = \begin{pmatrix} F_{n+1}(x) - F_n(x) & F_n(x) \\ xF_n(x) & F_n(x) - F_{n-1}(x) \end{pmatrix}$.

By substituting this matrix in $Q^a \cdot Q^{b-1} = Q^c \cdot Q^{d-1}$, and considering (1,2) entrances of the result, it is obtained:

$$F_a(x)F_b(x) - F_c(x)F_d(x) = (-1)[F_{a-1}(x)F_{b-1}(x) - F_{c-1}(x)F_{d-1}(x)]$$

Now, by applying the same process r times, identity (12) is obtained. $\square$

Corollary 8 (d'Ocagne's identity). *For $n \leqslant m$ integers*:

$$F_{n+1}(x)F_m(x) - F_n(x)F_{m+1}(x) = (-1)^{n-1}F_{m-n}(x)$$

Proof. It is enough to do $a = n + 1$, $b = m$, $c = n$, $d = m + 1$ and $r = n - 1$ in Eq. (12). $\square$

Binet's formula (4) allows us to express the sum of the first n polynomials in an easy way. See [21, Proposition 8] for a proof.

Proposition 9 (*Sum of the first n* polynomials).

$$\sum_{i=1}^{n} F_i(x) = \frac{F_{n+1}(x) + F_n(x) - 1}{x}$$

3.1. Expression of x^n as a function of the Fibonacci polynomials

Note first, that the equations for the Fibonacci polynomials may be written in matrix form as $F = B \cdot X$, where $F = (F_1(x), F_2(x), F_3(x), \dots)^T$, $X = (1, x, x^2, x^3, \dots)^T$, and B is the lower triangular matrix with entrances the coefficients appearing in the expansion of the Fibonacci polynomials in increasing powers of x:

$$B = \begin{pmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 1 & 0 & 1 &&&&& \\ 0 & 2 & 0 & 1 &&&& \\ 1 & 0 & 3 & 0 & 1 &&& \\ 0 & 3 & 0 & 4 & 0 & 1 && \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 & \\ 0 & 4 & 0 & 10 & 0 & 6 & 0 & 1 \end{pmatrix}$$

Note that in matrix B the non-zero entrances build precisely the diagonals of the Pascal triangle and the sum of the elements in the same row gives the classical Fibonacci sequence. In addition, matrix B is invertible, and

$$B^{-1} = \begin{pmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ -1 & 0 & 1 &&&&& \\ 0 & -2 & 0 & 1 &&&& \\ 2 & 0 & -3 & 0 & 1 &&& \\ 0 & 5 & 0 & -4 & 0 & 1 && \\ -5 & 0 & 9 & 0 & -5 & 0 & 1 & \\ 0 & -14 & 0 & 14 & 0 & -6 & 0 & 1 \end{pmatrix}$$

and, therefore, x^n may be written as linear combination of Fibonacci polynomials:

$$\begin{aligned} 1 &= F_1(x) \\ x &= F_2(x) \\ x^2 &= F_3(x) - F_1(x) \\ x^3 &= F_4(x) - 2F_2(x) \\ x^4 &= F_5(x) - 3F_3(x) + 2F_1(x) \\ x^5 &= F_6(x) - 4F_4(x) + 5F_2(x) \\ x^6 &= F_7(x) - 5F_5(x) + 9F_3(x) - 5F_1(x) \\ x^7 &= F_8(x) - 6F_6(x) + 14F_4(x) - 14F_2(x) \end{aligned}$$

These expansions are given in closed form in the following theorem, which is the version of the Zeckendorf's theorem for the Fibonacci polynomials. Zeckendorf's theorem establishes that every integer may be written in a unique way as sum of non-consecutive Fibonacci numbers: $n = \sum_{i=1}^{r} e_i F(i)$, where $e_i = 1$, or $e_i = 0$ and $e_i \cdot e_{i+1} = 0$ [30,31]. For the Fibonacci polynomials we have the following result.

Theorem 10. *For every integer $n \geqslant 1$, x^{n-1} may be written in a unique way as linear combination of the n first Fibonacci polynomials as*

$$x^{n-1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left[\binom{n}{i} - \binom{n}{i-1} \right] F_{n-2i}(x) \tag{13}$$

where $\binom{n}{-1} = 0$.

Proof. By induction. Eq. (13) is trivially true for $n = 1$. Let us suppose the Eq. (13) is true for every integer less or equal than $n - 1$. Then

$$x^{n-2} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \left[\binom{n-1}{i} - \binom{n-1}{i-1} \right] F_{n-1-2i}(x)$$

where by multiplying by x and having in mind that $xF_{n-2i-1}(x) = F_{n-2i}(x) - F_{n-2i-2}(x)$ then from the second term all the terms cancel and Eq. (13) for n is obtained. □

Corollary 11. *Every polynomial* $P_n(x) = \sum_{i=0}^{n} a_i x^i$ *may be written in a unique way as linear combination of Fibonacci polynomials.*

3.2. Fibonacci polynomials and Catalan's triangle

The coefficients, in absolute value, of the successive powers x^n written by increasing order of Fibonacci polynomials may be written in a 2-triangle, as the Pascal 2-triangle. See Table 5.

Note that in this 2-triangle the first double antidiagonal is of 1's. Then each element of any row is the sum of the two elements of the previous row: that on the same place on the row and the preceding one. That is, the recurrence law is $a_n(i) = a_{n-1}(i) + a_{n-1}(i-1)$. Finally for all even row, at the end is added the same last element of the previous row. Also note that if the diagonals of this 2-triangle are written as the rows of a new triangle we get the so-called Catalan's triangle. See Table 6.

Catalan's triangle shows many properties. Note, for example, that the sum of the elements in a row is equal to the last element of the following row, and the first diagonal coincides with the second one and gives precisely the Catalan's sequence: $C_1 = \{1, 1, 2, 5, 14, 42, 132, 429, \ldots\}$. This sequence is the second column of matrix B^{-1} without considering the sings of the entrances.

Catalan's numbers may be obtained by the following formulas: $C_n = \frac{1}{n+1}\binom{2n}{n}$ or $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$.

Table 5
Another Pascal 2-triangle

0.							1						
1.							1						
2.						1		1					
3.						1		2					
4.					1		3		2				
5.					1		4		5				
6.				1		5		9		5			
7.				1		6		14		14			
8.			1		7		20		28		14		
9.			1		8		27		48		42		
10.		1		9		35		75		90		42	
11.		1		10		44		110		165		132	
12.	1		11		54		154		275		297		132
13.	1		12		65		208		429		572		429

Table 6
Catalan's triangle

0.							1						
1.						1		1					
2.					1		2		2				
3.				1		3		5		5			
4.			1		4		9		14		14		
5.		1		5		14		28		42		42	
6.	1		6		20		48		90		132		132

Since given two sequences $\{a_n\}$ and $\{b_n\}$ their convolution is the new sequence $\{c_n\}$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ [32], diagonal $C_i = \{1, i, \ldots\}$ is precisely the convolution of order i of Catalan's sequence C_1, so $C_i = \otimes^i C_1$. Also it is verified $C_i = C_{i-1} \otimes C_1$.

Last formula for Catalan's numbers says that each Catalan's number is obtained by the convolution of the preceding terms.

A simple explanation of the Catalan's numbers is that they give the number of possible partitions in triangles of a regular polygon, such that the triangles have their vertices in the polygon vertices. Also, Catalan's numbers give the number of monotone paths in the plane between points $(0,0)$ and (n,n). A path between points $(0,0)$ and (n,n) is monotone if consists on unitary segments from left to right and from bottom to top and never goes through the diagonal.

4. Derivative of the Fibonacci polynomials

In this section we shall study the sequences obtained by deriving the Fibonacci polynomials. Then by giving to variable x integer values $x = 1, 2, 3, \ldots$ many integer sequences are generated. Several properties of these sequences, and the relations between the Fibonacci polynomials and their derivatives are proved.

4.1. Polynomials obtained by deriving the Fibonacci polynomials

By deriving the Fibonacci polynomials it is obtained:

$$F'_1(x) = 0$$
$$F'_2(x) = 1$$
$$F'_3(x) = 2x$$
$$F'_4(x) = 3x^2 + 2$$
$$F'_5(x) = 4x^3 + 6x$$
$$F'_6(x) = 5x^4 + 12x^2 + 3$$
$$F'_7(x) = 6x^5 + 20x^3 + 12x$$
$$F'_8(x) = 7x^6 + 30x^4 + 30x^2 + 4$$

Considering only the coefficients of the derivative polynomials, they may be written again in a double triangle, as shown in Table 7.

An interesting property of this triangle is that by summing up the elements of two alternate rows and dividing the result by the order of the second row the Fibonacci number corresponding to that order is obtained. For example, by

Table 7
The *derivative* Pascal 2-triangle

1.						1					
2.						2					
3.					3		2				
4.					4		6				
5.				5		12		3			
6.				6		20		12			
7.			7		30		30		4		
8.			8		42		60		20		
9.	9			56		105		60			5
10.	10			72		168		140			30

Table 8
Triangle from the quasi-diagonals and the Pascal triangle

1.					1												1			
2.				2		2										1		1		
3.			3		6		3			$\rightarrow$					1		2		1	
4.		4		12		12		4						1		3		3		1
5.	5		20		30		20		5			1			4		6		4		1

Table 9
The *derivative* scaled by the antidiagonal Pascal 2-triangle

1.					1				
2.					2				
3.				3		1			
4.				4		3			
5.			5		6		1		
6.			6		10		4		
7.		7		15		10		1	
8.		8		21		20		5	
9.	9		28		35		15		1
10.	10		36		56		35		6

summing up the fifth row and the seventh row and dividing by 7 it is obtained $F_7(1) = 13$. This result will be proved later in Proposition 17.

Note that by rearranging the terms of the derivative Pascal 2-triangle writing down by rows the terms appearing in the respective quasi-diagonal, a new triangle arises (see Table 8 (left)). If the ith file of this triangle is divided by i, again the classical Pascal triangle appears (see Table 8 (right)).

Note also that by dividing by i each element of the ith antidiagonal in Table 7, the triangle obtained is shown in Table 9, which actually is the same Pascal 2-triangle after deleting the first antidiagonal (see Table 3).

If we derive again, we will get the second derivatives of the Fibonacci polynomials:

$$F''_1(x) = 0$$
$$F''_2(x) = 0$$
$$F''_3(x) = 2$$
$$F''_4(x) = 6x$$
$$F''_5(x) = 12x^2 + 6$$
$$F''_6(x) = 20x^3 + 24x$$
$$F''_7(x) = 30x^4 + 60x^2 + 12$$
$$F''_8(x) = 42x^5 + 120x^3 + 60x$$

From where we may write the coefficients again in triangular shape. See Table 10 (left). In this triangle, by dividing each element of the ith antidiagonal between $i(i+1)$ the triangle shown Table 10 (right) results. Note that this last triangle is precisely the Pascal 2-triangle without its two first antidiagonals.

As before, by rearranging the terms of the second derivative Pascal 2-triangle in the same form as indicated previously, we get again the classical Pascal triangle (see Table 11 (right)).

And this procedure follows for any successive derivative of Fibonacci polynomials.

Table 10
The *second derivative* Pascal 2-triangle

1.			2						1		
2.			6						3		
3.		12		6				6		1	
4.		20		24		→		10		4	
5.	30		60		12		15		10		1
6.	42		120		60		21		20		5

Table 11
Triangle from the 2nd-derivative triangle and the Pascal triangle

1.					2										1				
2.				6		6								1		1			
3.			12		24		12			→			1		2		1		
4.		20		60		60		20				1		3		3		1	
5.	30		120		180		120		30		1		4		6		4		1

4.2. Numerical sequences obtained from the derivatives of Fibonacci polynomials

Different numerical sequences are obtained by simple substitution of variable x for an integer into the derivative of Fibonacci polynomials. For example, for the first derivative, we get

$$\{F'_n(1)\} = \{0, 1, 2, 5, 10, 20, 38, 71, 130, 235, \ldots\}$$
$$\{F'_n(2)\} = \{0, 1, 4, 14, 44, 131, 376, 1052, 2888, 7813, \ldots\}$$
$$\{F'_n(3)\} = \{0, 1, 6, 29, 126, 516, 2034, 7807, 29382, \ldots\}$$
$$\{F'_n(4)\} = \{0, 1, 8, 50, 280, 1475, 7472, 36836, 178000, \ldots\}$$

Sequence $\{F'_n(1)\}$ is studied in [23], in which it is numbered as sequence $A001629$. In that site is underlined that the kth term of the sequence represents the number subsets of $\{1, 2, \ldots, k-1\}$ with no consecutive integers. For example: $a(5) = 10$ because there are 10 subsets of $\{1, 2, 3, 4\}$ that have no consecutive elements: $\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$ (Emeric Deutsch (deutsch@duke.poly.edu), December 10, 2003). Sequence $\{F'_n(2)\}$ is also studied in [23], and numbered as sequence $A006645$. Sequences $\{F'_n(1)\}$ and $\{F'_n(2)\}$ are the only ones appearing in [23]. It should be noted, however, that these two integer sequences, along with the other in the list verify the general formulas and identities which we will show later on.

The same scheme followed with the list of polynomials $\{F'_n(x)\}$ can be observed in order to obtain other numerical sequences from the list of mth derivative polynomial. So, for example, from the second derivative, follows:

$$\{F''_n(1)\} = \{0, 0, 2, 6, 18, 44, 102, 222, 466, 948, \ldots\}$$
$$\{F''_n(2)\} = \{0, 0, 2, 12, 54, 208, 732, 2424, 7684, 23568, \ldots\}$$
$$\{F''_n(3)\} = \{0, 0, 2, 18, 114, 612, 2982, 13626, 59474, \ldots\}$$
$$\{F''_n(4)\} = \{0, 0, 2, 24, 198, 1376, 8652, 50928, 286036, \ldots\}$$

While for the third derivative, we get

$$\{F'''_n(1)\} = \{0, 0, 0, 6, 24, 84, 240, 630, 1536, 3564, \ldots\}$$
$$\{F'''_n(2)\} = \{0, 0, 0, 6, 48, 264, 1200, 4860, 18192, \ldots\}$$
$$\{F'''_n(3)\} = \{0, 0, 0, 6, 72, 564, 3600, 20310, 105408, \ldots\}$$
$$\{F'''_n(4)\} = \{0, 0, 0, 6, 96, 984, 8160, 59580, 399264, \ldots\}$$

Proposition 12 (Asymptotic behaviour of the quotient of consecutive terms). *If* $\sigma = \frac{k+\sqrt{k^2+4}}{2}$, *then*

$$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \lim_{n\to\infty} \frac{F'_{n+1}(x)}{F'_n(x)} = \lim_{n\to\infty} \frac{F''_{n+1}(x)}{F''_n(x)} = \cdots = \sigma$$

However, the rate of convergence to the corresponding metallic mean decreases when the order of derivation increases. For example, for $k = 3$ and $n = 9$, and without considering the initial null terms is obtained: $\frac{F_{10}(3)}{F_9(3)} = \frac{42837}{12970} = 3.3027$, $\frac{F'_{10}(3)}{F'_9(3)} = \frac{108923}{29382} = 3.7071$, $\frac{F''_{10}(3)}{F''_9(3)} = \frac{250812}{59474} = 4.2171$, $\frac{F'''_{10}(3)}{F'''_9(3)} = \frac{514956}{105408} = 4.8853$, when $\sigma_3 = \frac{3+\sqrt{13}}{2} = 3.3027$.

4.3. First relation between the derivative sequence and the Fibonacci sequence

Proposition 13.

$$F'_n(x) = \frac{nF_{n+1}(x) - xF_n(x) + nF_{n-1}(x)}{x^2 + 4} \tag{14}$$

Proof. By deriving into the Binet's formula (10) it is obtained:

$$F'_n(x) = n\frac{\sigma^{n-1} - (-\sigma)^{-n-1}}{\sigma + \sigma^{-1}}\sigma' - \frac{\sigma^n - (-\sigma)^{-n}}{(\sigma + \sigma^{-1})^2}(1 - \sigma^{-2})\sigma'$$

being $\sigma = \frac{x+\sqrt{x^2+4}}{2}$, and therefore $\sigma' = \frac{\sigma}{\sigma+\sigma^{-1}}$, $1 - \sigma^{-2} = \frac{x}{\sigma}$, and then

$$F'_n(x) = n\frac{\sigma^n + (-\sigma)^{-n}}{(\sigma + \sigma^{-1})^2} - \frac{\sigma^n - (-\sigma)^{-n}}{\sigma + \sigma^{-1}} \cdot \frac{x}{(\sigma + \sigma^{-1})^2} = n\frac{\sigma^n + (-\sigma)^{-n}}{(\sigma + \sigma^{-1})^2} - \frac{xF_n(x)}{(\sigma + \sigma^{-1})^2}$$

On the other hand, $F_{n+1}(x) + F_{n-1}(x) = \frac{\sigma^{n+1}-(-\sigma)^{-n-1}}{\sigma+\sigma^{-1}} + \frac{\sigma^{n-1}-(-\sigma)^{-n+1}}{\sigma+\sigma^{-1}} = \frac{\sigma}{\sigma^2+1}[\sigma^{n-1}(\sigma^2+1) - (-\sigma)^{-n-1}(1+\sigma^2)] = \sigma^n + (-\sigma)^{-n}$.

From where, after some algebra Eq. (14) is obtained. □

Since $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ Eq. (14) may be re-written as $F'_n(x) = \frac{x(n-1)F_n(x)+2nF_{n-1}(x)}{x^2+4}$. In particular, if $x = 1$ results $F'_n(1) = \frac{(n-1)F_n+2nF_{n-1}}{5}$ [4].

4.4. Expression of the derivative of the Fibonacci polynomials

Deriving in Eq. (8) it is obtained:

$$F'_{n+1}(x) = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}(n-2i)\binom{n-i}{i}x^{n-1-2i} \quad \text{for } n \geqslant 1 \tag{15}$$

while $F'_1(x) = 0$.

And, in the same way, an explicit formula for any derivative may be obtained. For example for the second derivative of the Fibonacci polynomials we have

$$F''_{n+1}(x) = \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}(n-2i)(n-1-2i)\binom{n-i}{i}x^{n-2-2i} \quad \text{for } n \geqslant 2 \tag{16}$$

while $F''_n(x) = 0$ for $n = 1, 2$.

4.5. Second relation between the derivative sequence and the Fibonacci sequence

Sequence $\{F'_n(x)\}_{n\in N}$ may be obtained by the self-convolution of the x-Fibonacci sequence, as the following Proposition establishes.

Proposition 14 (*The derivative of the Fibonacci polynomials and the convolved Fibonacci polynomials*).

$$F'_1(x) = 0, \text{ and } F'_n(x) = \sum_{i=1}^{n-1}F_i(x)F_{n-i}(x) \quad \text{for } n > 1 \tag{17}$$

Proof (*By induction*). For $n = 2$ is trivial, since $F'_2(x) = F_1(x)F_1(x) = 1$. Let us suppose that the formula is true for every polynomial $F'_k(x)$ with $k \leqslant n$. Then, $F'_{n-1}(x) = \sum_{i=1}^{n-2}F_i(x)F_{n-1-i}(x); F'_n(x) = \sum_{i=1}^{n-1}F_i(x)F_{n-i}(x)$ By deriving in equation: $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ and using previous expression it is obtained:

$$\begin{aligned} F'_{n+1}(x) &= F_n(x) + xF'_n(x) + F'_{n-1}(x) = F_n(x) + x\sum_{i=1}^{n-1}F_i(x)F_{n-i}(x) + \sum_{i=1}^{n-2}F_i(x)F_{n-1-i}(x) \\ &= F_n(x) + xF_{n-1}(x)F_1(x) + \sum_{i=1}^{n-2}xF_i(x)F_{n-i}(x) + \sum_{i=1}^{n-2}F_i(x)F_{n-1-i}(x) \\ &= F_n(x) + xF_{n-1}(x) + \sum_{i=1}^{n-2}F_i(x)[xF_{n-i}(x) + F_{n-1-i}(x)] = F_n(x)F_1(x) + F_{n-1}(x)F_2(x) + \sum_{i=1}^{n-2}F_i(x)F_{n+1-i}(x) \\ &= \sum_{i=1}^{n}F_i(x)F_{n+1-i}(x) \qquad \square \end{aligned}$$

It should be noted the similarity between the expression (17) for the derivative of the Fibonacci polynomials and the convolution formula for the Catalan's numbers. Also by using Eq. (14) together with Eq. (17) yields

$$\sum_{i=1}^{n-1}F_i(x)F_{n-i}(x) = \frac{x(n-1)F_n(x) + 2nF_{n-1}(x)}{x^2+4} \quad \text{for } n > 1$$

which for $x = 1$ gives the corresponding formula for the Fibonacci numbers, see [32, Eq. (7.61)].

4.6. Third relation between the derivative sequence and the Fibonacci sequence

Proposition 15.

$$F'_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i (n-2i) F_{n-2i}(x) \quad \text{for } n \geqslant 1 \tag{18}$$

Proof (*By induction*). For $n = 1$ the formula is true since $F'_2(x) = \sum_{i=0}^{0}(-1)^i(1-2i)F_{1-2i}(x) = 1$. Let us suppose that Eq. (18) is held till the derivative of the nth Fibonacci polynomial. Let us also suppose that n is an even integer, that is $n = 2p$. Then $F'_{2p+1}(x) = \sum_{i=0}^{p-1}(-1)^i(2p-2i)F_{2p-2i}(x)$ and $F'_{2p}(x) = \sum_{i=0}^{p-1}(-1)^i(2p-1-2i)F_{2p-1-2i}(x) = \sum_{i=0}^{p-1}(-1)^i(2p-2i)F_{2p-1-2i}(x) + \sum_{i=0}^{p-1}(-1)^{i+1}F_{2p-1-2i}(x)$. Now, since $F_{2p+2}(x) = xF_{2p+1}(x) + F_{2p}(x)$, $F'_{2p+2}(x) = F_{2p+1}(x) + xF'_{2p+1}(x) + F'_{2p}(x)$ and Eq. (18) is verified for $n = 2p + 2$ after some algebra. The case for n an odd integer is analogously checked. □

For instance, $F'_6(2)$ is given by $F'_6(2) = \sum_{i=0}^{2}(-1)^i(5-2i)F_{5-2i}(2) = 5F_5(2) - 3F_3(2) + F_1(2) = 5 \cdot 29 - 3 \cdot 5 + 1 = 131$.

4.7. Fourth relation between the derivative sequence and the Fibonacci sequence

In this section we shall establish a new relation between the Fibonacci polynomials and their derivatives.

Proposition 16.

$$F_n(x) = \frac{1}{n}[F'_{n+1}(x) + F'_{n-1}(x)] \tag{19}$$

Proof. By Eq. (8) $F_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}$, and $F_{n-1}(x) = \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i} x^{n-2-2i}$, and, hence their sum results:

$$F_{n+1}(x) + F_{n-1}(x) = x^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i} + \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i} x^{n-2-2i} = x^n + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-i}{i} + \binom{n-1-i}{i-1}\right] x^{n-2i}$$

Now, taking into account that

$$\binom{n-i}{i} + \binom{n-1-i}{i-1} = \binom{n-1-i}{i-1}\left(\frac{n-i}{i} + 1\right) = \binom{n-1-i}{i-1} \cdot \frac{n}{i}$$

we can deduce that $F_{n+1}(x) + F_{n-1}(x) = x^n + n\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-i}{i-1} \frac{1}{i} x^{n-2i}$, where by deriving it is obtained: $F'_{n+1}(x) + F'_{n-1}(x) = n \cdot x^{n-1} + n\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-i}{i-1} \frac{n-2i}{i} x^{n-1-2i}$ and hence $\frac{F'_{n+1}(x) + F'_{n-1}(x)}{n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} x^{n-1-2i} = F_n(x)$. □

Note that, from Eq. (19) $F'_{n+1}(x) = nF_n(x) - F'_{n-1}(x)$, and then, given the x-Fibonacci sequence $\{F_n(x)\}_{n \in N}$ and having in mind that $F'_1(x) = 0$ the derivative sequence $\{F'_n(x)\}_{n \in N}$ may be easily obtained.

4.8. Generating function for the derivative polynomials

Function $f_k(x) = \frac{x}{1-kx-x^2}$ is the generating function of the k-Fibonacci polynomials [20]. Therefore, a simple way for obtaining the generating function for the derivative of the Fibonacci polynomials is by deriving function $f_k(x)$ with respect to variable k. In this form $f_k^{(r)}(x) = r!(\frac{t}{1-xt-t^2})^{r+1}$ is the generating function of the rth derivative of the Fibonacci polynomials.

Another equivalent way for obtaining the generating function of the Fibonacci polynomials consists in the use of the convolution theorem, since the derivative Fibonacci polynomials may be seen as the self-convolution of the Fibonacci polynomials [32]. So, the generating function of the derivative sequence $\{F'_{n+1}(x)\}$ is the square of the generating function of sequence $\{F_{n+1}(x)\}$, that is $G_n^{(1)}(x) = G_n^2(x)$. From where, by deriving, it can be obtained the generating function of the sequence $\{F''_{n+1}(x)\}_{n \in N}$, and so on.

4.9. A recurrence relation into the derivative sequence

Proposition 17.

$$F_{n+1}^{(r)}(x) = \begin{cases} 0, & \text{if } n < r \\ r!, & \text{if } n = r \\ \frac{1}{n-r}[nx \cdot F_n^{(r)}(x) + (n+r)F_{n-1}^{(r)}(x)], & \text{if } n > r \end{cases} \tag{20}$$

It should be noted that all non-null terms on the rth derivative are obtained from the first non-null term, which is $r!$, and therefore all the terms are multiple of $r!$. As a consequence, sequence $\{F_n^{(r)}(k)\}_{n\in N}$ is of the form $\{F_n^{(r)}(k)\}_{n\in N} = r!\{\overbrace{0,\cdots,0}^{r},1,(r+1)k,\ldots\}$.

4.10. The integral of the Fibonacci polynomial

From Eq. (19) is straightforwardly obtained the following result.

Proposition 18.

$$\int_0^x F_n(x)\,\mathrm{d}x = \frac{1}{n}(F_{n+1}(x) + F_{n-1}(x) - F_{n+1}(0) - F_{n-1}(0))$$

If n is even, then $F_{n+1}(0) = F_{n-1}(0) = 1$ and, in this case $\int_0^x F_n(x)\,\mathrm{d}x = \frac{1}{n}(F_{n+1}(x) + F_{n-1}(x) - 2)$. If n is odd, then $F_{n+1}(0) = F_{n-1}(0) = 0$ and, in this case $\int_0^x F_n(x)\,\mathrm{d}x = \frac{1}{n}(F_{n+1}(x) + F_{n-1}(x))$.

5. Conclusions

The k-Fibonacci polynomials are the natural extension of the k-Fibonacci numbers. Many of their properties have been straightforwardly proven. In particular, the derivatives of these polynomials have been presented as convolution of the k-Fibonacci polynomials. This fact allows us to present in an easy way a family of integer sequences in a new and direct form.

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