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The metallic ratios as limits of complex valued transformations

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Abstract

We study the presence of the metallic ratios as limits of two complex valued transformations. These complex variable functions are introduced and related with the two geometric antecedents for each triangle in a particular triangle partition, the four-triangle longest-edge (4TLE) partition. In this way, the fractality of a geometric diagram for the classes of dissimilar generated triangles is also explained.

1. Introduction

One of the simplest and more studied integer sequence is the Fibonacci sequence [1–9]: $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, \ldots\}$ wherein each term is the sum of the two preceding terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. On the other hand the ratio of two consecutive Fibonacci numbers converges to the golden mean, or golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$, which appears in modern research in many fields from Architecture, Nature and Art [10–17] to theoretical physics [18–24]. For instance, El Naschie proposes a quantum golden field theory (QGFT), based on the golden mean binary. In this theory the classical quantum field theory is included [25].

The paper presented here was originated for the astonishing presence not only of the *golden ratio* but also of the general *metallic ratios* [11] in a recursive partition of triangles in the context of the finite element method and triangular refinements.

1.1. Grid generation and triangles

Grid generation and, in particular, the construction of 'quality' grids is a major issue in both geometric modeling and engineering analysis [26–29]. Many of these methods employ forms of local and global triangle subdivision and seek to maintain well shaped triangles. The four-triangle longest-edge (4TLE) partition is constructed by joining the midpoint of the longest edge to the opposite vertex and to the midpoints of the two remaining edges [30,31]. The two subtriangles with edges coincident with the longest edge of the parent are similar to the parent. The remaining two subtriangles form a similar pair that, in general, are not similar to the parent triangle. We refer to such new triangle shapes as 'dissimilar' to those preceding. The iterative partition of obtuse triangles systematically improves the triangles

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in the sense that the sequence of smallest angles monotonically increases, while the sequence of largest angles monotonically decreases in an amount (at least) equal to the smallest angle of each iteration [27,31].

In this paper, we use the concept of antecedent of a (normalized) triangle to deduce a pair of complex variable functions. These functions, in matrix form, allow us to study the fractality of a geometric diagram which arises at studying the sequence of dissimilar triangles generated by iterative application of the 4TLE partition to any initial triangle [29].

2. Normalized triangles, antecedents and complex valued functions

Since we are interested in the shape of the triangles, each triangle is scaled to have the longest edge of unit length. In this form, each triangle is represented for the three vertices: $(0,0),(1,0)$ and $z=(x,y)$. Since the two first vertices are the extreme points of the longest edge, the third vertex is located inside two bounding exterior circular arcs of unit radius, as shown in Fig. 1. In the following, for any triangle t, the edges and angles will be respectively denoted in decreasing order $r_1 \geqslant r_2 \geqslant r_3$, and $\gamma \geqslant \beta \geqslant \alpha$.

Definition 1. The longest-edge (LE) partition of a triangle t_0 is obtained by joining the midpoint of the longest edge of t_0 with the opposite vertex (Fig. 2a). The four-triangle longest-edge (4TLE) partition is obtained by joining the midpoint of the longest edge to the opposite vertex and to the midpoints of the two remaining edges (see Fig. 2b).

In the 4TLE scheme, subdivision leads to subtriangles that are similar to some previous parent triangles in the refinement tree so generated. Other subtriangles may result that are not in such similarity classes yet and we refer to these as new dissimilar triangles. We define the class $\mathscr{C}_n$ as the set of triangles for which the application of the 4TLE partition produces exactly n dissimilar triangles.

Let us begin by describing a Monte Carlo computational experiment used to visually distinguish the classes of triangles by the number of dissimilar triangles generated by the 4TLE partition. We proceed as follows: (1) Select a point within the mapping domain comprised by the horizontal segment and by the two bounding exterior circular arcs. This point (x,y) defines the apex of a target triangle. (2) For this selected triangle, 4TLE refinement is successively applied as long as a new dissimilar triangle appears. This means that we recursively apply 4TLE and stop when the shapes of new generated triangles are the same as those already generated in previous refinement steps. (3) The number of such refinements to reach termination defines the number of dissimilar triangles associated with the initial triangle and this numerical value is assigned to the initial point (x,y) chosen. (4) This process is progressively applied to a large sample of triangles (points) uniformly distributed over the domain. (5) Finally, we graph the respective values of dissimilar triangles in a corresponding color map to obtain the result in Fig. 3.

Definition 2. (4TLE left and right antecedents) A given triangle t_{n+1}, has two (different) triangles t_n, denoted here as left and right antecedents, whose 4TLE partition produces triangle t_{n+1}.

As an example, triangle t_{n+1} in the diagram with vertices $(0,0),(1,0)$ and z in Fig. 4a, has left antecedent t_n with vertices $z,(0,0),$ and $z+1$ in Fig. 4b, and right antecedent t_n with vertices $z,(1,0),$ and $z-1$ in Fig. 4c.

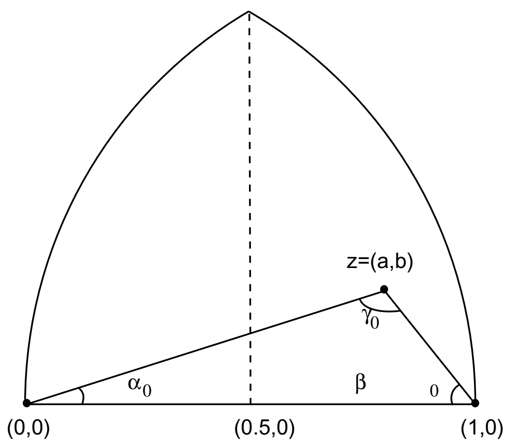

Fig. 1. Diagram for representing shape triangles.

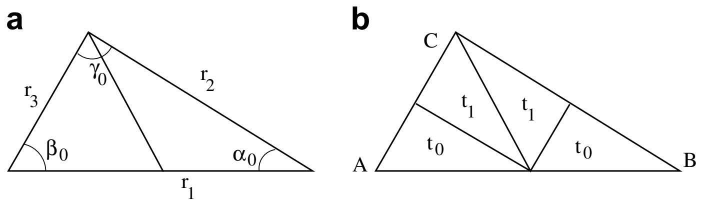

Fig. 2. (a) LE partition of triangle t_0, (b) 4TLE partition of triangle t_0.

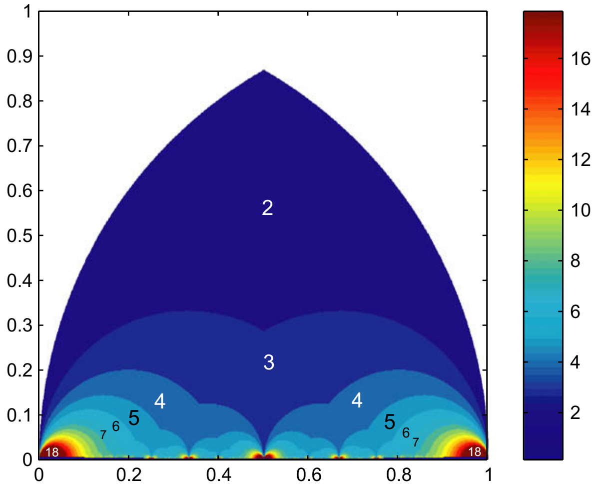

Fig. 3. Subregions for dissimilar triangle classes generated by Monte Carlo simulation.

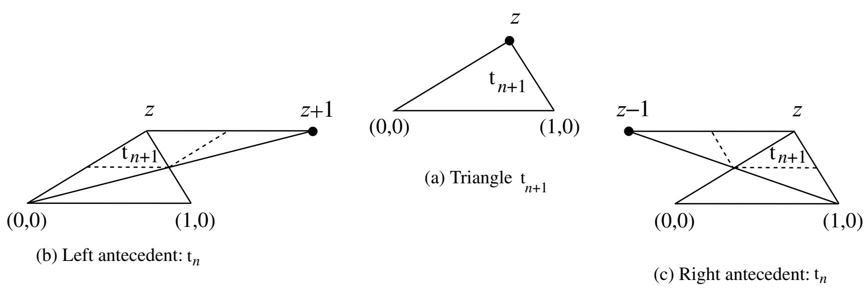

Fig. 4. Two antecedents for the 4TLE partition of triangle t_n.

Theorem 3. *The relation between the apex of a given triangle z in the right half of the diagram and the apices of its left and right antecedents may be mathematically expressed by the maps* $f_{\mathrm{L}}(z) = \frac{1}{\bar{z}+1}$, *and* $f_{\mathrm{R}}(z) = \frac{1}{2-z}$, *complex z. (See Figs. 5,6 where transformations* $f_L(z)$ *and* $f_R(z)$ *are deduced).*

Remark. Notice that for z having $\frac{1}{2} \leqslant \mathrm{Re}(z) \leqslant 1$ then $\mathrm{Re}(f_{\mathrm{L}}(z)) \leqslant \mathrm{Re}(f_{\mathrm{R}}(z))$ (and hence the 'left'/'right' terminology given to these complex functions).

Theorem 4. *The class separators determined experimentally in Fig. 3 may be generated mathematically as a recursive composition of left and right maps* $f_{\mathrm{L}}(z)$, *and* $f_{\mathrm{R}}(z)$.

Function f_{R} is a Moebius transformation (also homography or fractional linear transformation) [32,33], while function f_{L} is an anti-homography. Both transformations may be considered as maps of the completed complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself. f_{R} is a conformal map, and hence it preserves angles in magnitude and direction, and straight lines and circles are transformed into straight lines and circles. On the other hand, f_L is not conformal, but angles are preserved in magnitude and reversed in direction, as the complex conjugation. Also f_L takes circles to circles (straight lines count as circles of infinite radius) (see Fig. 5).

A general Moebius transformation $h(z) = \frac{az+b}{cz+d}$ can be defined by the matrix $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose elements are constant real numbers and its determinant is not null to avoid the constant transformation. In a similar way, an anti-homography presents the form $h(\bar{z}) = \frac{a\bar{z}+b}{c\bar{z}+d}$ and also has the same associated real matrix.

Moebius transformation f_R is obtained by the ordered application of an inversion $f_1(z) = \frac{1}{z}$, a rotation $f_2(z) = e^{\pi i}z = -z$ and a translation $f_3(z) = z - 2$. So, $f_R(z) = (f_3 \circ f_2 \circ f_1)(z) = f_3(f_2(f_1(z)))$. Therefore, if we note by R the matrix associated to f_R and by R_i the respective matrix associated to transformation f_i, then $R = R_3 \cdot R_2 \cdot R_1$ (see Fig. 6):

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Anti-homography f_L results as the product of the inversion $g_3(z) = \frac{1}{z}$, symmetry with respect to real axis $g_2(z) = \bar{z}$ and translation $g_1(z) = z + 1$. So, by noting with L, and L_i the matrices, respectively associated to f_L and g_i for $i = 1, 2, 3$, we have $L = L_3 \cdot L_2 \cdot L_1$, and so,

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 5. (1) *Both transformations f_R and f_L transform real numbers in real numbers.*

(2) *Both transformations transform upper half plane (and lower half plane) into itself.*

(3) *If $|\mathrm{Im}\,P| > 1$ then $|\mathrm{Im}\,f(P)| < |\mathrm{Im}\,P|$, for $f = f_R$ and $f = f_L$, so in this case $f(P)$ is closer to the real axis than P.*

Proof. (1) It is trivial.

(2) Let P be a point on the upper half plane $P = a + bi$ with $b \neq 0$ and consider function f_R. Then $f_R(P) = \frac{1}{2-a-bi} = \frac{2-a+bi}{(2-a)^2+b^2}$, and $\mathrm{Im}\,f_R(P) = \frac{b}{(2-a)^2+b^2}$ with the same sign that $\mathrm{Im}\,P$, so P and $f_R(P)$ are in the same half plane.

(3) If $|\mathrm{Im}\,P| = |b| > 1$, then $|\mathrm{Im}\,f_R(P)| = |\frac{b}{(2-a)^2+b^2}| \leqslant \frac{1}{|b|} < 1$. Note that $\mathrm{Im}\,f_L(P) = \frac{b}{(1+a)^2+b^2}$. □

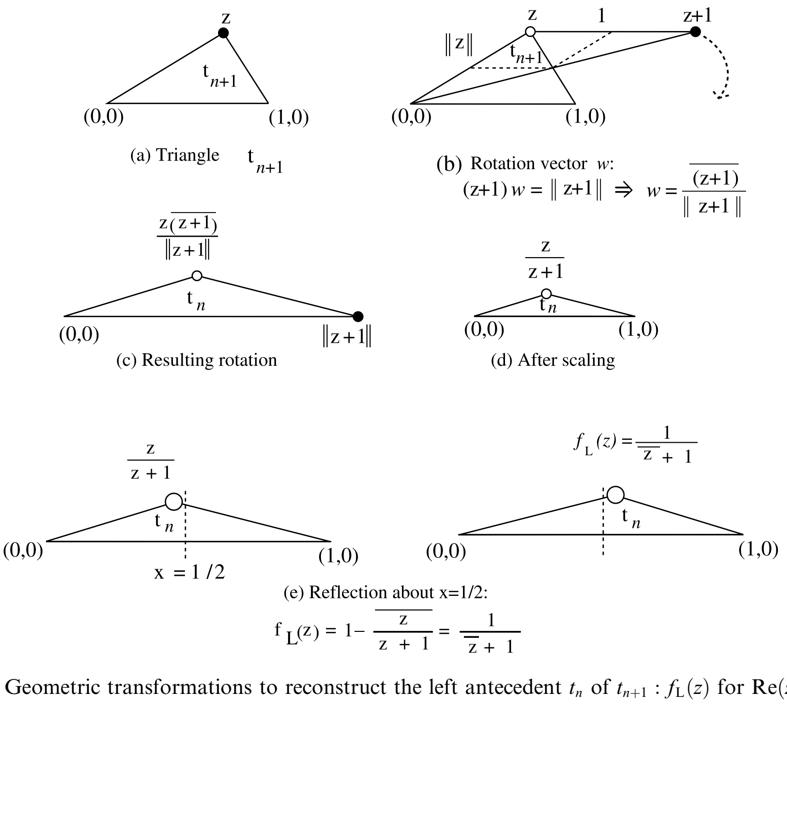

Fig. 5. Geometric transformations to reconstruct the left antecedent t_n of $t_{n+1} : f_L(z)$ for $\mathrm{Re}(z) \geqslant \frac{1}{2}$.

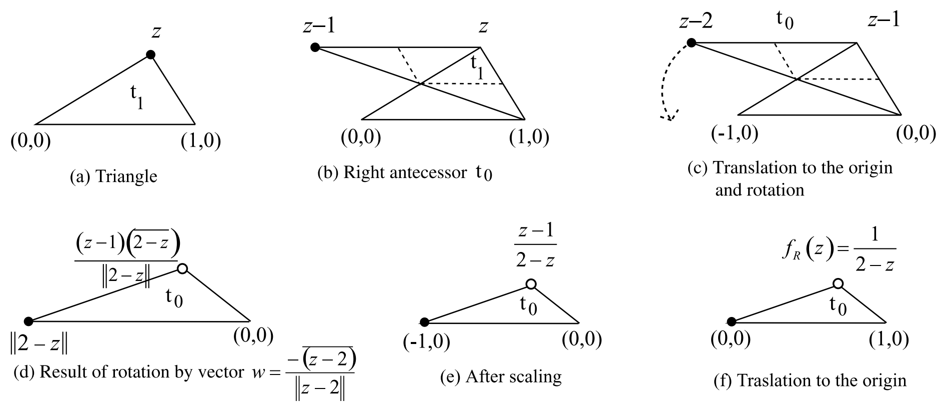

Fig. 6. Geometric transformations to reconstruct the right antecedent t_n of $t_{n+1} : f_R(z)$ for $\mathrm{Re}(z) \geqslant \frac{1}{2}$.

As a consequence, observe that the iterative application of functions f_R and f_L in any order converges to a real number.

Theorem 6. *The image by* f_R *of the second quadrant* $C_2 = \{z \in \mathbb{C},\ \mathrm{Re}(z) \leqslant 0, \mathrm{Im}(z) \geqslant 0\}$ *is the half circle* $\{z \in \mathbb{C},\ z \leqslant \frac{1}{2} + \frac{1}{2}e^{i\theta},\ 0 \leqslant \theta \leqslant \pi\}$.

Proof. Let $P = a + bi$ be a point in C_2. Then $f_R(P) = \frac{2-a+bi}{(2-a)^2+b^2}$, and if d denotes the distance from $f_R(P)$ to point $(1/2, 0)$, then $d^2 = (\frac{2-a}{(2-a)^2+b^2} - \frac{1}{2})^2 + (\frac{b}{(2-a)^2+b^2})^2 = \frac{a-1}{(2-a)^2+b^2} + \frac{1}{4} = \frac{1}{4} \cdot \frac{a^2+b^2}{(2-a)^2+b^2} \leqslant \frac{1}{4}$. □

Corollary 7. *The image of the iterated application of* f_R *to the completed complex plane* $\overline{\mathbb{C}}$ *is the circle* $\{z \in \mathbb{C}, z \leqslant \frac{1}{2} + \frac{1}{2}e^{i\theta},\ 0 \leqslant \theta \leqslant 2\pi\}$.

Theorem 8. $f_R(P) = f_L(P)$ *if and only if* $\mathrm{Re}(P) = \frac{1}{2}$.

Proof. If $P = \frac{1}{2} + bi$ then $f_R(P) = \frac{2}{3-2bi} = f_L(P)$.

Reciprocally, let $P = a + bi$ and suppose that $f_R(P) = f_L(P)$. Then $\frac{2-a}{(2-a)^2+b^2} + \frac{b}{(2-a)^2+b^2}i = \frac{a+1}{(1+a)^2+b^2} + \frac{b}{(1+a)^2+b^2}i$, and therefore, $2 - a = 1 + a$, so $a = \frac{1}{2}$. □

2.1. Fixed points

We study here the fixed points of transformations f_R and f_L, and, as a consequence, of any combination of these functions.

Note that every Moebius function of the form $f(z) = \frac{az+b}{cz+d}$ has at most two fixed points since equation $f(z) = z$ is equivalent to a second degree equation.

We call attractive fixed point to the fixed point of Moebius transform $f(z)$ which is limit of the sequence $\{f^n(z)\}_{n \in \mathbb{N}}$ for any initial complex z. In other case, the fixed point of f will be called repulsive.

However, one of the fixed points of the anti-homography f_L has the particularity that if P is a point in a neighborhood of the fixed point, then the sequence $\{f^n(P)\}_{n \in \mathbb{N}}$ converges to it but it does pivoting around it. That is, approaching to it and going far away, alternatively. For this reason, this fixed point will be called isolated fixed point, since it is only the limit of the constant sequence.

In conclusion, one of the fixed points of transformation f_L is isolated and the other one is attractive as it will be shown later on.

A transformation of the complex plane is parabolic if the determinant of its associated matrix is 1 and the square of its trace is 4. A necessary and sufficient condition for a transformation to be parabolic is that it has a unique fixed point. In addition, any non-parabolic transformation presents two fixed points. The transformation is elliptic if its trace verifies $\mathrm{trace}^2 < 4$, and it is hyperbolic if $\mathrm{trace}^2 > 4$.

The Moebius transformation f_{R} is parabolic since the determinant of its associated matrix is 1 and its trace is 2. However, although matrix L also holds these properties the anti-homography f_{L} is not parabolic because it is not a Moebius transformation. In fact, f_{L} is hyperbolic.

Theorem 9. *$z = 1$ is the unique fixed point of f_{R}.*

Proof. Since $f_{\mathrm{R}}(z) = \frac{1}{2-z}$ it is enough to solve equation $z = \frac{1}{2-z}$. □

Note that, as a consequence, f_{R} is parabolic, and hence the iterative application of f_{R} with any initial point converges to the fixed point $z = 1$.

Theorem 10. *$-\phi = -\frac{1+\sqrt{5}}{2}$ and $\frac{1}{\phi} = \frac{\sqrt{5}-1}{2}$ are the two fixed points of f_{L}.*

Proof. It is enough to solve the equation $\frac{1}{\bar{z}+1} = z$. ϕ is the well-known golden ratio. □

In the next section, we will study the iterated application of the functions f_{R} and f_{L} to any point of the complex plane $\mathbb{C}$. We will prove that the anti-homography f_{L} is related with the classical Fibonacci sequence $\{F_n\}$.

3. Transformations f_{R}^n and f_{L}^n

We consider here the applications obtained after n compositions of transformations f_{R} and f_{L}, that is f_{R}^n and f_{L}^n.

Theorem 11. *f_{R}^n is defined by $f_{\mathrm{R}}^n(z) = \frac{(-n+1)z+n}{-nz+n+1}$.*

Proof. By using the associated matrices, the proof is straightforwardly obtained since the associated matrix to f_{R} is $R = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, then $R^n = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix}$ is the associated matrix to f_{R}^n, as it can be easily verified by induction. □

Corollary 12. *Sequence $\{f_{\mathrm{R}}^n(z)\}_{n\in N}$ converges to the fixed point $z = 1$.*

Proof. It is trivial since for any initial point $z \in \mathbb{C}$ $\lim_{n\to\infty} \frac{(-n+1)z+n}{-nz+n+1} = 1$. □

Note that for any $n \geqslant 1, \mathrm{tr}(R^n) = 2$ and the pole of $(f_{\mathrm{R}})^n$ is $z_\infty = \frac{n+1}{n}$.

Although f_{L} is not a Moebius transformation, $(f_{\mathrm{L}})^{2k}$ does, for $k = 1, 2, 3, \ldots$, while for odd exponent $(f_{\mathrm{L}})^{2k+1}$ is an anti-homography. For this reason, the iterative application of f_{L} will be written as function of variable w, where $w = z$ if n is even, and $w = \bar{z}$ if n is odd.

Theorem 13. *The iterative application of f_{L} is defined by $f_{\mathrm{L}}^n(z) = \frac{F_{n-1}w+F_n}{F_n w+F_{n+1}}$, where F_i is the ith Fibonacci number.*

Proof. Since the associated matrix to f_{L} is $L = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, then $L^n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$ is the associated matrix to f_{L}^n, as it can be easily verified by induction. □

Corollary 14. *Sequence $\{f_{\mathrm{L}}^n(z_0)\}_{n\in N}$ converges to the fixed point $z = \frac{1}{\phi}$ if the initial value $z_0 \neq -\phi$, while $f_{\mathrm{L}}^n(-\phi) = -\phi$ is an isolated fixed point, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

Proof. Let us suppose $z \neq -\phi$, then by taking limits in the expression of $f_{\mathrm{L}}^n : \lim_{n\to\infty} f_{\mathrm{L}}^n(z) = \lim_{n\to\infty} \frac{F_{n-1}w+F_n}{F_n w+F_{n+1}} = \lim_{n\to\infty} \frac{w+\frac{F_n}{F_{n-1}}}{\frac{F_n}{F_{n-1}}w+\frac{F_{n+1}}{F_{n-1}}} = \frac{w+\phi}{\phi w+\phi^2} = \frac{w+\phi}{\phi(w+\phi)} = \frac{1}{\phi}$, where it has been taking into account that $\lim_{k\to\infty} \frac{F_k}{F_{k-1}} = \phi$ and also $\lim_{k\to\infty} \frac{F_k+1}{F_{k-1}} = \lim_{k\to\infty} \frac{F_k+1}{F_k} \frac{F_k}{F_{k-1}} = \phi^2$.

And hence, if $w + \phi \neq 0$, then $\lim_{n\to\infty} f_{\mathrm{L}}^n(z) = \frac{1}{\phi}$, while the other fixed point is $\lim_{n\to\infty} f_{\mathrm{L}}^n(-\phi) = -\phi$. □

Note that the iterated application of transformation f_{L} to any initial point gives a convergent sequence to its unique fixed point $z_0 = 1$. On the other hand, the iterated application of transformation f_{R} to any initial point gives a sequence of points with real parts alternating around the fixed point $z_0 = -\phi$.

3.1. Rate of convergence of sequences $\{f_R^n(z_0)\}_{n\in N}$ and $\{f_L^n(z_0)\}_{n\in N}$

Let $\{a_n\}$ be a convergent numerical sequence such that $\lim_{n\to\infty} a_n = L$. The rate of convergence of $\{a_n\}$ is usually given by the quotient $|\frac{a_n-L}{a_{n-1}-L}|$, i.e. the minor this quotient is the faster the sequence converges.

In our case, we choose as initial point $z_0 = \frac{\phi}{2}$ which is the midpoint of the two attractive fixed points of the two transformations f_R and f_L, 1 and $\frac{1}{\phi}$, respectively. In this situation for the sequences $\{f_R^n(z_0)\}_{n\in N}$ and $\{f_L^n(z_0)\}_{n\in N}$ the rates of convergence are $\left|\frac{f_R(f_R(\frac{\phi}{2}))-1}{f_R(\frac{\phi}{2})-1}\right| = 0.861803$ and $\left|\frac{f_L(f_L(\frac{\phi}{2}))-1}{f_L(\frac{\phi}{2})-1}\right| = 0.398016$, which implies that sequence $\{f_L^n(z_0)\}_{n\in N}$ converges faster than $\{f_R^n(z_0)\}_{n\in N}$ does.

3.2. Other properties of f_L^n

$\mathrm{trace}(L^n) = F_{n-1} + F_{n+1}$ and its pole is $z_\infty = -\frac{F_{n+1}}{F_n}$. In addition, by the property of previous paragraph, and taking into account Cassini's identity:

$(\mathrm{trace}\,L^n)^2 = (F_{n-1}+F_{n+1})^2 = F_{n-1}^2 + F_{n+1}^2 + 2F_{n-1}F_{n+1} = F_{n-1}^2 + F_{n+1}^2 + 2F_n^2 + 2(-1)^{n+1}$. So $(\mathrm{trace}\,L^n)^2 > 4$ for $n = 1, 2, 3, \ldots$ and hence these transformations are of hyperbolic type having two different fixed points as we already knew.

3.2.1. Inverse transformations

To obtain the inverse transformations of f_R and f_L it is sufficient to find the inverse matrices of the respective associated matrices, having in mind that the multiplying factor corresponding to the inverse of the determinant, is not necessary for obtaining the inverse transformation because the associated matrix represents a quotient in which this multiplying factor acts as a multiplying factor both for the numerator and for the denominator, and hence, it can be eliminated from the inverse matrix [33]. Therefore, in the sequel, R^{-1} and L^{-1} are for the matrices associated to the inverse transformations f_R^{-1} and f_L^{-1} which are not exactly the same that the inverse of the associated matrices R and L.

Note, that the fixed points of a given transformation and its inverse are the same, although their characters are not necessarily equal. In this way, an attractive fixed point for a transformation may be a repulsive fixed point for its inverse transformation, and vice versa. In fact, the attractive fixed point of transformation f_L is an isolated fixed point of f_L^{-1} and reciprocally.

So, transformations f_R^{-1} and f_L^{-1} are respectively represented by the matrices $R^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ and $L^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Then, $f_R^{-1}(z) = \frac{2z-1}{z}$ and $f_L^{-1}(z) = \frac{-\bar{z}+1}{\bar{z}}$.

Following the same argument as before it can be easily proved that:

(a) $f_R^{-n}(z) = \frac{-(n+1)z+n}{-nz+n-1}$, and therefore, $\lim_{n\to\infty} f_R^{-n}(z) = 1$.

(b) $f_L^{-n}(z) = \frac{F_{n+1}w-F_n}{-F_nw+F_{n-1}}$, where $w = z$ if n is even, and $w = \bar{z}$ if n is odd. Therefore, $\lim_{n\to\infty} f_L^{-n}(z) = -\phi$, if $z \neq \phi$.

In the next section, se will study the product of transformations f_R and f_L and their properties.

4. Product of transformations f_R and f_L

It is clear that not only iterative applications of single transformations f_R and f_L may be considered, but also any combination of (finite) products of such transformations. This is the focus of this section.

Proposition 15. *The composition of transformations f_R and f_L is not commutative, but it is associative.*

Proof. Since the product of matrices R and L is not commutative because $R \cdot L = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, while $L \cdot R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 3 \end{pmatrix}$, then the composition of transformations f_R and f_L cannot be commutative.

On the other hand, since matrix multiplication is associative, the same is true for the composition of three or more transformations f_R and f_L [33].

About the inverse of the product of these transformations, it is easy to check that:

(a) $(f_{\rm R}\circ f_{\rm L})^{-1}=f_{\rm L}^{-1}\circ f_{\rm R}^{-1}$, $(f_{\rm L}\circ f_{\rm R})^{-1}=f_{\rm R}^{-1}\circ f_{\rm L}^{-1}$.
(b) $f_{\rm R}^{-1}\circ f_{\rm L}^{-1}\neq f_{\rm L}^{-1}\circ f_{\rm R}^{-1}$. □

4.1. Poles of these transformations

Given a Moebius transformation or an anti-homography, its fixed points and the corresponding poles to this transformation and to its inverse function are the opposite vertices of a parallelogram [33,34] called characteristic parallelogram of the transformation. Since the fixed points and also the poles of transformations $f_{\rm R}$ and $f_{\rm L}$ and of their integer powers are real numbers, the fixed points are the extreme points of a segment in the real axis. The midpoint of this segment is also the midpoint of the segment which extreme points are the pole of the transformation and the pole of its inverse. That is, if z_0 and z_0' are the fixed points of one of these transformations, say f, and z_∞ is its pole and Z_∞ is the pole of its inverse transformation f^{-1}, then $\frac{z_0+z_0'}{2}=\frac{z_\infty+Z_\infty}{2}$.

Moreover, since the fixed points and the poles of these transformations are real numbers, the poles z_∞, Z_∞ are into the segment with extreme points z_0 and z_0' at the same distance from its center. Hence the sum of the fixed points is equal to the sum of its pole plus the pole of its inverse transformation: $z_0+z_0'=z_\infty+Z_\infty$.

Finally, since the normalized matrix of a transformation is $A=\begin{pmatrix} Z_\infty & -z_0z_0' \\ 1 & -z_\infty \end{pmatrix}$. For example, it is easy to find that the fixed points of transformation $f_{\rm L}\circ f_{\rm L}\circ f_{\rm R}$ are $\frac{3\pm\sqrt{3}}{2}$, and its pole is $\frac{5}{2}$. Therefore, the associated matrix with determinant 1 is $A=\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ 1 & -\frac{5}{2} \end{pmatrix}$, which is similar to matrix $L\cdot L\cdot R=\begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}$, in the sense that both generate the same transformation.

Note here that since the product of these transformations is not commutative in general, it makes no sense to try to find a formula for certain number of applications $f_{\rm R}$ and $f_{\rm L}$ without any order. Instead, below we study the iterated applications of two compositions $f_{\rm R}^{k-1}\circ f_{\rm L}$ and $f_{\rm L}^{k-1}\circ f_{\rm R}$.

5. Transformations of the type $(f_{\rm R}^{k-1}\circ f_{\rm L})^n$

The composition of two of such functions has as associated matrix the pro-duct of the matrices associated to the two initial transformations. Similarly, any particular combination of transformations $f_{\rm R}$ and $f_{\rm L}$ is determined by the product of the corresponding matrices in the same order. For instance, transformation $(f_{\rm R}\circ f_{\rm L}(z))=f_{\rm R}(f_{\rm L}(z))=f_{\rm R}(\frac{1}{\bar z+1})=\frac{1}{2-\frac{1}{\bar z+1}}=\frac{\bar z+1}{2\bar z+1}$ could be given more easily by the matrix product

$$R\cdot L=\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Therefore, from now on, we will substitute the use of the transformations $f_{\rm R}$ and $f_{\rm L}$ by the use of the respective associated matrices R and L.

Let us find the product $R^{k-1}L$ associated to the composition $f_{\rm R}^{k-1}\circ f_{\rm L}$ which will be used below. It is easy to prove that $R^{k-1}=\begin{pmatrix} -k+2 & k-1 \\ -k+1 & k \end{pmatrix}$ for all $k\geqslant 1$, and so $R^{k-1}\cdot L=\begin{pmatrix} -k+2 & k-1 \\ -k+1 & k \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} k-1 & 1 \\ k & 1 \end{pmatrix}$.

5.1. k-Fibonacci numbers

k-Fibonacci numbers have been recently introduced and studied in different contexts [35–39]. We relate here these numbers with the complex variable functions $f_{\rm R}$ and $f_{\rm L}$.

Definition 16. For any integer number $k\geqslant 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$F_{k,0}=0,\ F_{k,1}=1,\ \text{and}\ F_{k,n+1}=kF_{k,n}+F_{k,n-1}\quad\text{for } n\geqslant 1.$$

Particular cases of the previous definition are:

- If $k=1$, the classical Fibonacci sequence is obtained: $F_0=0,\ F_1=1,$ and $F_{n+1}=F_n+F_{n-1}$ for $n\geqslant 1$: $\{F_n\}_{n\in\mathbb{N}}=\{0,1,1,2,3,5,8,\dots\}$.
- If $k=2$, the Pell sequence appears: $P_0=0,\ P_1=1,$ and $P_{n+1}=2P_n+P_{n-1}$ for $n\geqslant 1:\{P_n\}_{n\in\mathbb{N}}=\{0,1,2,5,12,29,70,\dots\}$.
- If $k=3$, the following sequence appears: $F_{3,0}=0,\ F_{3,1}=1,$ and $F_{3,n+1}=3F_{3,n}+F_{3,n-1}$ for $n\geqslant 1:\{F_{3,n}\}_{n\in\mathbb{N}}=\{0,1,3,10,33,109,\dots\}$.

The relation between matrix $R^{k-1}\cdot L$ and the kth Fibonacci sequence is given by the following proposition.

Proposition 17. *For any integer $n\geqslant 1$ holds*:

$$(R^{k-1}\cdot L)^n=\begin{pmatrix} F_{k,n+1}-F_{k,n} & F_{k,n}\\ F_{k,n+1}-F_{k,n-1} & F_{k,n}+F_{k,n-1}\end{pmatrix}. \tag{1}$$

Proof (*By induction*). For $n=1$:

$$R^{k-1}\cdot L=\begin{pmatrix} k-1 & 1\\ k & 1\end{pmatrix}=\begin{pmatrix} F_{k,2}-F_{k,1} & F_{k,1}\\ F_{k,2}-F_{k,0} & F_{k,1}+F_{k,0}\end{pmatrix},$$

since $F_{k,0}=0,\ F_{k,1}=1,$ and $F_{k,2}=k$.

Let us suppose that the formula is true for $n-1$:

$$(R^{k-1}\cdot L)^{n-1}=\begin{pmatrix} F_{k,n}-F_{k,n-1} & F_{k,n-1}\\ F_{k,n}-F_{k,n-2} & F_{k,n-1}+F_{k,n-2}\end{pmatrix}.$$

Then

$$(R^{k-1}\cdot L)^n=(R^{k-1}\cdot L)^{n-1}(R^{k-1}\cdot L)=\begin{pmatrix} F_{k,n}-F_{k,n-1} & F_{k,n-1}\\ F_{k,n}-F_{k,n-2} & F_{k,n-1}+F_{k,n-2}\end{pmatrix}\cdot\begin{pmatrix} k-1 & 1\\ k & 1\end{pmatrix}$$
$$=\begin{pmatrix} (k-1)F_{k,n}+F_{k,n-1} & F_{k,n}\\ kF_{k,n} & F_{k,n}+F_{k,n-1}\end{pmatrix}=\begin{pmatrix} F_{k,n+1}-F_{k,n} & F_{k,n}\\ F_{k,n+1}-F_{k,n-1} & F_{k,n}+F_{k,n-1}\end{pmatrix}. \qquad \square$$

Recall that last relation means that

$$(f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}})^n(z)=\frac{(F_{k,n+1}-F_{k,n})w+F_{k,n}}{(F_{k,n+1}-F_{k,n-1})w+F_{k,n}+F_{k,n-1}},$$

where $w=z$ if n is odd, and $w=\bar{z}$ if n is even.

5.2. Fixed points and pole of transformation $f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}}$

For obtaining the fixed points of this transformation it is enough to calculate $\lim_{n\to\infty}(f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}})^n(z)$ or, equivalently, $\lim_{n\to\infty}\frac{(F_{k,n+1}-F_{k,n})w+F_{k,n}}{(F_{k,n+1}-F_{k,n-1})w+F_{k,n}+F_{k,n-1}}$. Note that $\lim_{n\to\infty}\frac{F_{k,n+r}}{F_{k,n}}=\sigma^r$, where σ is the positive characteristic root of the polynomial associated to the recurrence relation, $\sigma=\frac{k+\sqrt{k^2+4}}{2}$ [35–37]. Then by dividing numerator and denominator of the previous limit between $F_{k,n-1}$, and having in mind that the fixed points of these transformations are real numbers, it is obtained: $\lim_{n\to\infty}(f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}})^n(z)=\frac{(\sigma^2-\sigma)w+\sigma}{(\sigma^2-1)w+\sigma+1}=\frac{\sigma((\sigma-1)w+1)}{(\sigma+1)((\sigma-1)w+1)}=\frac{\sigma}{\sigma+1}$ in the case $z\neq-\sigma$, while if $z=-\sigma$, then $\lim_{n\to\infty}(f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}})^n(-\sigma)=-\sigma$ and therefore, $z=-\sigma$ is the unique attractive fixed point.

Note that the pole of each transformation $(f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}})^n$ is precisely $z(k,n)=-\frac{1}{k}(1+\frac{F_{k,n-1}}{F_{k,n}})$ which tends to the non-attractive fixed point for $n\to\infty$.

Particular cases are:

- If $k=1$, the classical Fibonacci sequence is obtained: $F_0=0,\ F_1=1,$ and $F_{n+1}=F_n+F_{n-1}$ for $n\geqslant 1$, and therefore, the associated matrix to the transformation $(f_{\mathrm{R}}^{k-1}\circ f_{\mathrm{L}})^n=f_{\mathrm{L}}^n$ is $L^n=\begin{pmatrix} F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}$, which underlines the relation between anti-homography f_{L} and the classical Fibonacci sequence. Note that matrix L^n is very similar to the Fibonacci matrix Q defined by $Q = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}$, and then $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ [4].

The two fixed points are z_1 and z_2, being $z_1 = \frac{\phi}{\phi+1} = \frac{\phi}{\phi^2} = \frac{1}{\phi} = \sigma_1^{-1} = -\frac{1-\sqrt{5}}{2}$ which is the limit point of the sequence $\{f_L^n(z)\}_{n\in\mathbb{N}}$ if $z \neq -\phi$ and hence z_1 is attractive; $z_2 = -\frac{1}{\phi} = \sigma_1$ if $z = -\phi$ which is the repulsive fixed point.

- If $k = 2$, the Pell sequence appears: $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geqslant 1$. The associated matrix is $(RL)^n = \begin{pmatrix} P_n + P_{n-1} & P_n \\ 2P_n & P_n + P_{n+1} \end{pmatrix}$. Its fixed points are $z_1 = \frac{\sigma_2}{\sigma_2+1} = \frac{1+\sqrt{2}}{2+\sqrt{2}} = \frac{1}{\sqrt{2}}$ and $z_2 = -\sigma_2 = -1-\sqrt{2}$, where $\sigma_2 = 1+\sqrt{2}$ is the silver ratio [11]. Note that sequence $\{(f_R \circ f_L)^n(z)\}_{n\in\mathbb{N}}$ converges to z_1 for all $z \neq \sigma_2$ and it converges to z_2 if $z = \sigma_2$. The pole of each iteration is $z(n) = -\frac{1}{2}(1 + \frac{P_{n-1}}{P_n})$ and the sequence of poles results $\{-\frac{1}{2}, -\frac{3}{4}, -\frac{7}{10}, -\frac{17}{24}, \ldots \to -\frac{1}{\sqrt{2}}\}$.[1]
- If $k = 3$, the following sequence appears: $F_{3,0} = 0$, $F_{3,1} = 1$, and $F_{3,n+1} = 3F_{3,n} + F_{3,n-1}$ for $n \geqslant 1$: $\{F_n\}_{n\in\mathbb{N}} = \{0, 1, 3, 10, 33, 109, \ldots\}$. Let σ_3 denote the bronze ratio $\sigma_3 = \frac{3+\sqrt{13}}{2}$ [11]. The fixed points in this case are $\frac{\sigma_3}{\sigma_3+1} = \frac{1+\sqrt{13}}{6}$, and therefore, the sequence $\{(f_R^2 \circ f_L)^n(z)\}_{n\in\mathbb{N}}$ converges to $\frac{1+\sqrt{13}}{6}$.

Any recursive sequence obtained by iterative application of any combination of transformations f_R and f_L will converge to some real number which will be between the limit of sequence $\{f_L^n(z)\}_{n\in\mathbb{N}}, \frac{1}{\phi}$, and the limit of sequence $\{f_R^n(z)\}_{n\in\mathbb{N}}, 1$.

6. Transformation of the type $(f_L \circ f_R^{k-1})^n$

The associated matrix to this composition can be obtained as follows. In the previous section we have already obtained that

$$(R^{k-1} \cdot L)^{n+1} = \begin{pmatrix} F_{k,n+2} - F_{k,n+1} & F_{k,n+1} \\ kF_{k,n+1} & F_{k,n+1} + F_{k,n} \end{pmatrix}.$$

Consequently

$$R^{k-1}(L \cdot R^{k-1})^n L = \begin{pmatrix} F_{k,n+2} - F_{k,n+1} & F_{k,n+1} \\ kF_{k,n+1} & F_{k,n+1} + F_{k,n} \end{pmatrix}.$$

And, hence

$$(L \cdot R^{k-1})^n = (R^{k-1})^{-1} \begin{pmatrix} F_{k,n+2} - F_{k,n+1} & F_{k,n+1} \\ kF_{k,n+1} & F_{k,n+1} + F_{k,n} \end{pmatrix} L^{-1}.$$

And, therefore

$$(L \cdot R^{k-1})^n = \begin{pmatrix} k & -k+1 \\ k-1 & -k+2 \end{pmatrix} \begin{pmatrix} F_{k,n+2} - F_{k,n+1} & F_{k,n+1} \\ kF_{k,n+1} & F_{k,n+1} + F_{k,n} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(L \cdot R^{k-1})^n = \begin{pmatrix} F_{k,n+1} - (2k-1)F_{k,n} & kF_{k,n} \\ (3-2k)F_{k,n} & F_{k,n+1} + (k-1)F_{k,n} \end{pmatrix}.$$

For obtaining the fixed points it is sufficient to solve the equation

$$\frac{[F_{k,n+1} - (2k-1)F_{k,n}]w + kF_{k,n}}{(3-2k)F_{k,n}w + F_{k,n+1} + (k-1)F_{k,n}} = z,$$

considering that the fixed points will not depend on n. Since the fixed points are real numbers, this yields to the following equation:

$$[F_{k,n+1} - (2k-1)F_{k,n}]x + kF_{k,n} = (3-2k)F_{k,n}x^2 + [F_{k,n+1} + (k-1)F_{k,n}]x$$

from which we obtain the fixed points.

[1] It is worthy to be noted that the sequence of the numerators of sequence $\{z(n)\}$ without sign is referenced in [40] as $A001333$, and the sequence of denominators is referenced as $A052452$.

Table 1
Symbolic sequences, associated matrices, fixed point and poles

Symbolic sequence	Matrix	Attractive fixed point	Isolated fixed point	Pole
R^n	$\begin{pmatrix} -n+1 & n \\ -n & n+1 \end{pmatrix}$	1		$1+\frac{1}{n}$
L^n	$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$	$-\frac{1}{\phi}$	$-\phi$	$-\frac{F_{n+1}}{F_n}$
$(R^{k-1}L)^n$	$\begin{pmatrix} F_{k,n+1}-F_{k,n} & F_{k,n} \\ F_{k,n+1}-F_{k,n-1} & F_{k,n}+F_{k,n-1} \end{pmatrix}$	$\frac{\sigma_k}{\sigma_k+1}$	$-\sigma_k$	$-\frac{1}{k}\left(1+\frac{F_{k,n-1}}{F_{k,n}}\right)$
$(LR^{k-1})^n$	$\begin{pmatrix} F_{k,n+1}-(2k-1)F_{k,n} & kF_{k,n} \\ (3-2k)F_{k,n} & F_{k,n+1}+(k-1)F_{k,n} \end{pmatrix}$	$\frac{k-1+\sigma_k^{-1}}{2k-3}$	$\frac{k-1-\sigma_k}{2k-3}$	$\frac{1}{2k-3}(k-1+\frac{F_{k,n+1}}{F_{k,n}})$
$(RLR)^n$	$\begin{pmatrix} F_{3,n-1}-F_{3,n} & 3F_{3,n} \\ -F_{3,n} & 4F_{3,n}+F_{3,n-1} \end{pmatrix}$	$\frac{5-\sqrt{13}}{2}$	$\frac{5+\sqrt{13}}{2}$	$4+\frac{F_{3,n-1}}{F_{3,n}}$
$(LRL)^n$	$\begin{pmatrix} 2a_n-a_{n-1} & a_n \\ 3a_n & 2a_n-a_{n-1} \end{pmatrix}$, where $a_n=\frac{s^n-s^{-n}}{s+s^{-1}}$ being $s=2+\sqrt{3}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$	$-\frac{2}{3}+\frac{a_{n-1}}{a_n}$

$z(k)=\frac{2-3k\pm\sqrt{k^2+4}}{2(3-2k)}$, being the attractive fixed point the positive value of $z(k)=\frac{k-1+\sigma_k^{-1}}{2k-3}$ and repulsive fixed point $z(k)=\frac{k-1-\sigma_k}{2k-3}$.

In addition, the pole of this transformation corresponds to the real number

$$z(k,n)=\frac{1}{2k-3}\left(k-1+\frac{F_{k,n+1}}{F_{k,n}}\right),$$

which yields to the non-attractive fixed point when $n\to\infty$.

Particular cases are:

- If $k=1$, we have the matrix: $L^n=\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$, associated to the transformation f_L^n, where F_n is the nth classical Fibonacci number. Its fixed points are $\frac{1}{\phi}=\frac{-1+\sqrt{5}}{2}$ (attractive) and $-\phi=-\frac{-1+\sqrt{5}}{2}$ (repulsive). The poles are the real numbers $z(n)=-\frac{F_{n+1}}{F_n}$, which of different values of n gives the sequence $\{-1,-2,-\frac{3}{2},-\frac{5}{3},-\frac{8}{5},\ldots\to-\phi\}$.[2]
- If $k=2$, the following matrix appears: $(L\cdot R)^n=\begin{pmatrix} P_n+P_{n-1} & 2P_n \\ -P_n & P_{n+1}+P_n \end{pmatrix}$, where P_n is for the nth classical Pell number. Its fixed points are $z_1=2-\sqrt{2}$ (attractive) and $z_2=2+\sqrt{2}$ (repulsive). The poles are $z(n)=1+\frac{P_{n+1}}{P_n}$ and the sequence of poles results $\{3,\frac{7}{2},\frac{17}{5},\frac{41}{12},\ldots\to1+\sigma_2=2+\sqrt{2}\}$.[3]

Evidently, the fixed points of these transformations are precisely the fixed points of their inverse applications, and also the midpoint of the fixed points coincides with the midpoint of the poles of the transform and the corresponding inverse transform.

7. Conclusions

With the aim to explain a geometrical diagram we have studied in this paper different combinations of two complex functions, an homography and anti-homography. The same process can be applied to similar functions that can appear

[2] The sequence of numerators (without sign) is the classical Fibonacci sequence $A000045$ in [40].
[3] The sequence of numerators is the sequence $A001333$ in [40].

in other scenarios. This study has been motivated by the arising of two complex valued maps to represent the two antecedents in an specific four-triangle partition.

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Annex. Examples

Here we show some examples of functions defined by a few combinations of transformations f_R and f_L. Table 1 shows several functions of this type, the associated matrix, its corresponding fixed points and the pole.

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