

## $k$ -Fibonacci sequences modulo $m$

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### Abstract

We study here the period-length of the  $k$ -Fibonacci sequences taken modulo  $m$ . The period of such cyclic sequences is known as Pisano period, and the period-length is denoted by  $\pi_k(m)$ . It is proved that for every odd number  $k$ ,  $\pi_k(k^2 + 4) = 4(k^2 + 4)$ .

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### 1. Introduction

Many references may be given for Fibonacci numbers and related issues [1–9]. As it is well-known Fibonacci numbers and Golden Section,  $\phi = \frac{1+\sqrt{5}}{2}$ , appear in modern research in many fields from Architecture, Nature and Art [10–16] to theoretical physics [17–21].

The paper presented here is focused on determining the length of the period of the sequences obtained by reducing  $k$ -Fibonacci sequences taking modulo  $m$ . This problem may be viewed in the context of generating random numbers, and it is a generalization of the classical Fibonacci sequence modulo  $m$ .  $k$ -Fibonacci numbers have been ultimately introduced and studied in different contexts [22–24] as follows:

**Definition 1.** For any integer number  $k \geq 1$ , the  $k$ th Fibonacci sequence, say  $\{F_{k,n}\}_{n \in \mathbb{N}}$  is defined recurrently by:

$$F_{k,0} = 0, F_{k,1} = 1, \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad \text{for } n \geq 1.$$

Note that if  $k$  is a real variable  $x$  then  $F_{k,n} = F_{x,n}$  and they correspond to the Fibonacci polynomials defined by:

$$F_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ xF_n(x) + F_{n-1}(x) & \text{if } n > 1 \end{cases}.$$

Particular cases of the previous definition are:

- If  $k = 1$ , the classical Fibonacci sequence is obtained:  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_{n+1} = F_n + F_{n-1}$  for  $n \geq 1$ :  $\{F_n\}_{n \in \mathbb{N}} = \{0, 1, 1, 2, 3, 5, 8, \dots\}$ .

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- If  $k = 2$ , the Pell sequence appears:  $P_0 = 0, P_1 = 1$ , and  $P_{n+1} = 2P_n + P_{n-1}$  for  $n \geq 1$ :  $\{P_n\}_{n \in \mathbb{N}} = \{0, 1, 2, 5, 12, 29, 70, \dots\}$ .
- If  $k = 3$ , the following sequence appears:  $F_{3,0} = 0, F_{3,1} = 1$ , and  $F_{3,n+1} = 3F_{3,n} + F_{3,n-1}$  for  $n \geq 1$ :  $\{F_{3,n}\}_{n \in \mathbb{N}} = \{0, 1, 3, 10, 33, 109, \dots\}$ .

From the definition of the  $k$ -Fibonacci numbers, the first of them are presented in Table 1, and from these expressions, one may deduce the value of any  $k$ -Fibonacci number by simple substitution on the corresponding  $F_{k,n}$ . For example, the seventh element of the 4-Fibonacci sequence,  $\{F_{4,n}\}_{n \in \mathbb{N}}$ , is  $F_{4,7} = 46 + 5 \cdot 44 + 6 \cdot 42 + 1 = 5473$ .

By doing  $k = 1, 2, 3, \dots$  the respective  $k$ -Fibonacci sequences are obtained:

$$\begin{aligned} \{F_{1,n}\}_{n \in \mathbb{N}} &= \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}, \\ \{F_{2,n}\}_{n \in \mathbb{N}} &= \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \dots\}, \\ \{F_{3,n}\}_{n \in \mathbb{N}} &= \{0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, \dots\}, \\ \{F_{4,n}\}_{n \in \mathbb{N}} &= \{0, 1, 4, 17, 72, 305, 1292, 5473, 23184, 98209, 416020, \dots\}. \end{aligned}$$

Some of the properties the  $k$ -Fibonacci sequences verify are summarized bellow, (see [22–24] for details of the proofs):

- [Binet’s formula]  $F_{k,n} = \frac{\sigma^n - (\sigma^{-1})^n}{\sigma - \sigma^{-1}}$ , where  $\sigma = \frac{k + \sqrt{k^2 + 4}}{2}$  is the positive root of the characteristic equation associated to the recurrence relation defining  $k$ -Fibonacci numbers:  $r^2 - kr - 1 = 0$ .
- [First combinatorial formula for the general term]

$$F_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-1-2i} (k^2 + 4)^i.$$

- [Second combinatorial formula for the general term]

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i}.$$

- [Catalan’s identity]  $F_{k,n-r} F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r} F_{k,r}^2$
- [Simson’s identity]  $F_{k,n-1} F_{k,n+1} - F_{k,n}^2 = (-1)^n$
- [d’Ocagne’s identity]  $F_{k,m} F_{k,n+1} - F_{k,m+1} F_{k,n} = (-1)^n F_{k,m-n}$
- [Sum of the first  $n$  terms]  $\sum_{i=1}^n F_{k,i} = \frac{1}{k} (F_{k,n+1} + F_{k,n} - 1)$
- [Sum of the first  $n$  even terms]  $\sum_{i=1}^n F_{k,2i} = \frac{1}{k} (F_{k,2n+1} - 1)$
- [Sum of the first  $n$  odd terms]  $\sum_{i=0}^n F_{k,2i+1} = \frac{1}{k} F_{k,2n+2}$
- [Generating function]  $f_k(x) = \frac{x}{1 - kx - x^2}$

## 2. Pisano periods for the $k$ -Fibonacci sequences

In this Section, the sequences obtained by considering the remainders of the  $k$ -Fibonacci sequences modulo  $m$ , being  $m$  an integer, are considered. For any fixed integer  $m$ , and varying the value of parameter  $k = 1, 2, \dots$  a new sequence is obtained by considering the remainders modulo  $m$ . The  $n$ th Pisano period, written  $\pi(n)$ , is the period with which the sequence of  $k$ -Fibonacci numbers, modulo  $n$  repeats [25].

Table 1  
The first  $k$ -Fibonacci numbers

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$F_{k,1} = 1$
$F_{k,2} = k$
$F_{k,3} = k^2 + 1$
$F_{k,4} = k^3 + 2k$
$F_{k,5} = k^4 + 3k^2 + 1$
$F_{k,6} = k^5 + 4k^3 + 3k$
$F_{k,7} = k^6 + 5k^4 + 6k^2 + 1$
$F_{k,8} = k^7 + 6k^5 + 10k^3 + 4k$

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Each Pisano period begins with a zero, the first of which corresponds to the remainder of  $F_{k,0} = 0 \pmod m$  and always ends with an “1” as it will be shown bellow. The period-length corresponding to the value  $m$  is the number of remainders inside a period and it is denoted  $\pi_k(m)$ . Since any period may be contained one or more zeros, the set of remainders between two consecutive zeros including the first zero, but not the second one will be called *chain*. Later on, it will be proved that all the chains have the same number of elements and that every Pisano period contains either 1, or 2 or 4 chains.

Let us denote  $\{F_{k,n} \pmod m\}_{n \in \mathbb{N}}$  the sequence of the remainders of the  $k$ -Fibonacci sequence modulo  $m$ .

**Theorem 2.**  $\{F_{k,n} \pmod m\}_{n \in \mathbb{N}}$  is a simple periodic sequence of period less than  $m2$ .

See [25,26] for a proof for classical Fibonacci sequence. The same argument applies to  $k$ -Fibonacci numbers.

For the sake of clarity in the exposition, we will write  $\{r_1, r_2, \dots\}$  the sequence of remainders  $\{F_{k,n} \pmod m\}_{n \in \mathbb{N}}$  and we will consider the  $m2$  pairs of the remainders  $(r_1, r_2), (r_2, r_3), \dots, (r_{p2}, r_{p2+1})$ . If each one of these pairs is different from any other, then one of them should be of the form  $(0, 1)$  since there are only  $m2$  pairs of numbers modulo  $m$ .

If  $(r_p, r_{p+1}) = (0, 1)$ , then evidently,  $(r_{p+1}, r_{p+2}) = (1, k)$ , and therefore the sequence of the remainders is periodic with period  $p$ .

On the other hand, if two of these pairs are the same, for example, if  $(r_i, r_{i+1}) = (r_j, r_{j+1})$  for  $i < j$ , then also and  $(r_{i-1}, r_i) = (r_{j-1}, r_j)$  and so on, until eventually  $(r_1, r_2) = (r_{j-i}, r_{j-i+1}) = (1, k)$ , and therefore the sequence is periodic with period  $j - i$ .

**Corollary 3.** If  $m > 1$  every Pisano period begins with  $0, 1, \dots$

**Theorem 4.** If the prime factorization of  $m$  is  $m = \prod p_i^{e_i}$ , then  $\pi_k(\text{lcm}(p_i^{e_i})) = \text{lcm}(\pi_k(p_i^{e_i}))$ .

**Proof.** Each  $\pi_k(p_i^{e_i})$  is the period-length of the corresponding sequence  $\{F_{k,n} \pmod{p_i^{e_i}}\}$ , that is, the sequence  $\{F_{k,m} \pmod{p_i^{e_i}}\}$  repeats only after blocks of length  $c\pi_k(p_i^{e_i})$ . On the other hand,  $\pi_k(m)$  is the period-length of the sequence  $\{F_{k,n} \pmod m\}$ , which implies that  $\{F_{k,n} \pmod{p_i^{e_i}}\}$  repeats after  $\pi_k(m)$  terms for all values of  $i$ , and since any such number gives a period of  $\{F_{k,n} \pmod m\}$ , we conclude that  $\pi_k(m) = \text{lcm}(\pi_k(p_i^{e_i}))$ .  $\square$

**Corollary 5.** If  $r|m$ , then  $\pi_k(r)|\pi_k(m)$ .

**Remark 6.**  $\pi_{2r+1}(2) = 3$ , and  $\pi_{2r}(2) = 2$ , for  $r = 1, 2, 3, \dots$

**Theorem 7.** If the first two chains of the Pisano sequence  $\{F_{k,m} \pmod m\}$  are  $0, a_1, a_2, \dots, a_{r-2}, a_{r-1}$ , and  $0, b_1, b_2, \dots, b_{r-1}$ , then  $b_{2i+1} = a_{r-(2i+1)}$  and  $b_{2i} = m - a_{r-2i}$ .

**Proof.** Considering that  $F_{k,r+1} = kF_{k,r} + F_{k,r-1}$  and that  $0 \equiv F_{k,r} \pmod m$  it is obtained that  $b_1 \equiv F_{k,r+1} \pmod m \equiv F_{k,r-1} \pmod m \equiv a_{r-1}$ .

Now, since  $F_{k,r+2} = (k^2 + 2)F_{k,r} - F_{k,r-2}$ , then  $b_2 = \text{remainder} \left( \frac{F_{k,r+2}}{m} \right) = -\text{remainder} \left( \frac{F_{k,r-2}}{m} \right) = m - a_{r-2}$ .

Finally, iterating previous reasoning the proof is done.  $\square$

As a direct consequence of previous theorem, if in a sequence of remainders of  $k$ -Fibonacci sequences modulo  $m$  appear consecutively 1,0, then 1 is the last value of a Pisano period, and 0 is the beginning of the following period.

Also, note that every Pisano period for  $m > 2$  has an even number of elements. This fact is in complete agreement with the adaptation of the  $k$ -Fibonacci sequences as proved in [25, Theorem 4] from where if  $m > 2$  then  $\pi_k(m)$  is an even number.

**Corollary 8.** If a chain has an even number of elements, then the Pisano period presents either one or two chains.

**Proof.** If the last element of the chain is 1, then the first element of the next chain is 0, and therefore the first chain is the Pisano period. In other case, if the last element of the chain is not 1, then the second element of the following chain after 0, is the same that the last but one of the first chain, and so on, till the last element of the second chain is the same that the second one of the first chain, that is 1, and the period finishes.  $\square$

**Corollary 9.** If a chain has an odd number of elements, then the Pisano period has four chains.

**Proof.** If the first chain has an even number of elements then the last element of the second chain is  $m - 1$  which coincides with the second element of the third chain and then its difference with  $m$  is the last element of the fourth chain. It is deduced, that the last element of the fourth chain is 1, where the Pisano period ends.  $\square$

**Corollary 10.** Every Pisano Period has either one, or two or four chains.

The main result of this paper is the following one.

**Theorem 11.** If  $k$  is an odd number, then  $\pi_k(k^2 + 4) = 4(k^2 + 4)$ .

**Proof.** From the first combinatorial formula for the general term of the  $k$ -Fibonacci sequence (Section 1), if  $n = k^2 + 4$ , then

$$F_{k,k^2+4} = \binom{k}{2}^{k^2+3} \sum_{i=0}^{\lfloor \frac{k^2+3}{2} \rfloor} \binom{k^2+4}{2i+1} k^{-2i} (k^2+4)^i$$

from where  $F_{k,k^2+4}$  is multiple of  $k^2 + 4$ . That is  $F_{k,k^2+4} \equiv 0 \pmod{k^2 + 4}$ , so this element is the first term of the next chain. And since  $k$  is odd also  $k^2 + 4$  is odd, and from Corollary 9 the length of the Pisano period is  $4(k^2 + 4)$ .  $\square$

In [27] is this proved that  $\pi(m)$  is less than or equal to  $6m$  for all  $m$  and that equality holds for infinitely many values of  $m$ . From Theorem 4, since  $\pi_k(2) = 3$  iff  $k$  is odd, following corollary results.

**Corollary 12.** For any odd number  $k$

$$\pi_k(2(k^2 + 4)) = 6(2(k^2 + 4)). \tag{1}$$

Notice that for every power of  $k^2 + 4$ , it holds that  $\pi_k((k^2 + 4)^r) = 4(k^2 + 4)^r$ , and therefore in any  $k$ -Fibonacci sequence there are infinite values of the modulo  $m$  for which Eq. (1) holds:

$$\pi_k(2(k^2 + 4)^r) = \pi_k(2) \cdot \pi_k(k^2 + 4)^r = \pi_k(2) \cdot (\pi_k(k^2 + 4))^r = 3 \cdot 4(k^2 + 4)^r = 6(2(k^2 + 4)^r).$$

**Remark 13.** If  $k$  is an odd number and  $k^2 + 4$  is composite, then 5 divides  $k^2 + 4$  and in the corresponding  $k$ -Fibonacci sequence  $\pi_k(5) = 4 \cdot 5$  and  $\pi_k\left(\frac{k^2+4}{5}\right) = 4\frac{k^2+4}{5}$ , and also for any combination of these factors. Note that  $k$  odd and  $k^2 + 4$  composite implies that  $k = 10r \pm 1$ . Then  $k = 10r \pm 1$  implies  $k^2 + 4 = 5a$  and the relation  $\pi_k(m) = 4m$  holds for every number  $m$  of the form  $m = 5^{n_1} a^{n_2}$  for  $n_1, n_2 \in \mathbf{N}$ .

For example, for  $k = 9$ ,  $k^2 + 4 = 85 = 5 \cdot 17$  and  $\pi_{29}(m) = 4m$  is true for any combination  $m = 5^{n_1} 17^{n_2}$  like 5, 17, 85, 25, 125, 289, 425, ...

**Theorem 14.** If  $k$  is an even number greater than or equal to 2, then  $\pi_k(2k) = 4$ . If  $k$  is an odd number greater than or equal to 3, then  $\pi_k(2k) = 6$ .

**Proof.** If  $k = 2r$ ,  $r \geq 1$ , then the  $k$ -Fibonacci sequence of polynomials has the form  $\{0, 1, 2r, (2r)2 + 1, (2r)3 + 2(2r), \dots\}$ . Having in mind that  $(2r)3 + 2(2r) = 4r(2r + 1)$ , the sequence of remainders modulo  $(4r)$  is  $\{0, 1, 2r, 1, 0, \dots\}$  which implies that chain  $\{0, 1, 2r, 1\}$  is also a Pisano period, and  $\pi_k(2k) = 4$ .

If  $k = 2r + 1$ ,  $r \geq 1$ , then the first  $k$ -Fibonacci polynomials are the following:

$$\begin{aligned} F_{k,0} &= 0, \\ F_{k,1} &= 1, \\ F_{k,2} &= 2r + 1, \\ F_{k,3} &= (2r + 1)2 + 1 = 4r2 + 4r + 2, \\ F_{k,4} &= (2r + 1)3 + 2(2r + 1) = 8r3 + 12r2 + 10r + 3, \\ F_{k,5} &= (2r + 1)4 + 3(2r + 1)2 + 1 = 16r4 + 32r3 + 36r2 + 20r + 5, \\ F_{k,6} &= (2r + 1)5 + 4(2r + 1)3 + 3(2r + 1) = 32r5 + 80r4 + 112r3 + 88r2 + 40r + 8. \end{aligned}$$

The sequence of the remainders modulo  $2k = 4r + 2$  is  $\{0, 1, 2r + 1, 2r + 2, 2r + 1, 1, 0, \dots\}$  and so the chain  $\{0, 1, 2r + 1, 2r + 2, 2r + 1, 1\}$  is the Pisano period and  $\pi_k(2k) = 6$ .  $\square$

**Theorem 15.** For any integers  $k$  and  $r \geq 1$ ,  $\pi_{r \cdot k}(k) = 2$ .

**Proof.** The  $(r \cdot k)$ -Fibonacci sequence is  $\{0, 1, rk, (rk)2 + 1, \dots\}$  and the remainder sequence modulo  $k$  is  $\{0, 1, 0, 1, \dots\}$ .  $\square$

Some examples of  $k$ -Fibonacci sequences for which the length of the Pisano period is 6 times the modulo are the following:

- For  $k = 1$ ,  $\pi(5) = 20$  and also  $\pi(10) = \pi(2)\pi(5) = 3(4 \cdot 5) = 60$ ,  $\pi(50) = \pi(2 \cdot 52) = \pi(2)\pi(52) = 3(4 \cdot 52) = 300$ ,  $\pi(250) = \pi(2 \cdot 53) = \pi(2)\pi(53) = 3(4 \cdot 53) = 1500$ , etc.
- For  $k = 3$ ,  $\pi_3(13) = 4 \cdot 13 = 52$  and also  $\pi_3(26) = \pi_3(2)\pi_3(13) = 3(4 \cdot 13) = 156$ ,  $\pi_3(2 \cdot 132) = \pi_3(2)\pi_3(132) = 3(4 \cdot 132) = 2028$ , etc.
- If  $k = 5$ ,  $\pi_5(29) = 4 \cdot 29 = 116$  and also  $\pi_5(2 \cdot 29) = \pi_5(2)\pi_5(29) = 3(4 \cdot 29) = 348$ ,  $\pi_5(2 \cdot 292) = \pi_5(2)\pi_5(292) = 3(4 \cdot 292) = 10092$ , etc.
- If  $k = 7$ ,  $\pi_7(53) = 4 \cdot 53 = 212$
- If  $k = 9$ ,  $k^2 + 4 = 85 = 5 \cdot 17$ , so  $\pi_9(5) = 4 \cdot 5$ ,  $\pi_9(17) = 4 \cdot 17$ ,  $\pi_9(85) = 4 \cdot 85$ ,  $\pi_9(25) = 4 \cdot 25$ , etc.

### 3. Pisano period-length sequences for the $k$ -Fibonacci sequences

For  $m = 1$  the only remainder is 0 and therefore the sequence of the lengths of the Pisano periods for any  $k$ -Fibonacci sequence should begin with 1. Next, the sequences of the Pisano period-lengths for the first five  $k$ -Fibonacci sequences are shown:

$$\begin{aligned} \{F_{1,n} \pmod m\} &= \{1, 3, 8, 6, 20, 24, 16, 12, 24, 60, 10, 24, 28, 48, 40, 24, 36, 24, \dots\}, \\ \{F_{2,n} \pmod m\} &= \{1, 2, 8, 4, 12, 8, 6, 8, 24, 12, 12, 8, 28, 6, 24, 16, 16, 24, 40, 12, \dots\}, \\ \{F_{3,n} \pmod m\} &= \{1, 3, 2, 6, 12, 6, 16, 12, 6, 12, 8, 6, 52, 48, 12, 24, 16, 6, 40, 12, \dots\}, \\ \{F_{4,n} \pmod m\} &= \{1, 2, 8, 2, 20, 8, 16, 4, 8, 20, 10, 8, 28, 16, 40, 8, 12, 8, 6, 20, \dots\}, \\ \{F_{5,n} \pmod m\} &= \{1, 3, 8, 6, 2, 24, 6, 12, 8, 6, 24, 24, 12, 6, 8, 24, 36, 24, 40, 6, \dots\}. \end{aligned}$$

Sequence  $\{F_{1,n} \pmod m\}$  is for the Pisano period-lengths corresponding to the classical Fibonacci sequence ( $k = 1$ ) [25] and is the only previous sequence appearing in [28] where it is listed as A00175. Sequence  $\{F_{2,n} \pmod m\}$  is for the Pisano period-lengths of corresponding to the Pell sequence ( $k = 2$ ).

#### 3.1. Periodic sequences of Pisano period-lengths of constant modulo

Table 2 shows the sequences obtained from the first 10  $k$ -Fibonacci sequences by doing  $m = 1, 2, \dots, 20$ . Each even element of a  $k$ -Fibonacci sequence is function of  $k$  while each odd element is of the form  $f(k) + 1$ , therefore for any  $k$ -Fibonacci sequence it holds that  $\pi_k(k) = \pi_k\{0, 1\} = 2$ .

On the other hand, taking into account that for every  $r \geq 1$  it is verified that  $\pi_r(m) = \pi_{k+r}(m)$ , then the sequence  $\{\pi_k(m)\}_{k \in \mathbb{N}}$  is periodic with period  $k$ . Also, since  $\pi_r(m) = \pi_{k-r}(m)$  for all  $r \geq 1$ , then in this sequence it is only necessary to find the first  $\lfloor \frac{k}{2} \rfloor$  lengths of the periods to obtain the corresponding sequence (see Table 3).

Table 2  
Pisano period-lengths for  $k$ -Fibonacci sequences

$k \backslash m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	3	8	6	20	24	16	12	24	60	10	24	28	48	40	24	36	24	18	60
2	1	2	8	4	12	8	6	8	24	12	12	8	28	6	24	16	16	24	40	12
3	1	3	2	6	12	6	16	12	6	12	8	6	52	48	12	24	16	6	40	12
4	1	2	8	2	20	8	16	4	8	20	10	8	28	16	40	8	12	8	6	20
5	1	3	8	6	2	24	6	12	8	6	24	24	12	6	8	24	36	24	40	6
6	1	2	2	4	20	2	16	8	6	20	24	4	6	16	20	16	36	6	8	20
7	1	3	8	6	12	24	2	12	24	12	10	24	6	6	24	24	8	24	40	12
8	1	2	8	2	12	8	16	2	24	12	8	8	12	16	24	4	68	24	18	12
9	1	3	2	6	20	6	6	12	2	60	24	6	28	6	20	24	68	6	18	60
10	1	2	8	4	2	8	16	8	24	2	10	8	52	16	8	16	8	24	18	4

**Example 16.** If  $k = 17$ , then  $\pi_{17}(17) = 2$  and we have only to find the lengths of the first eight Pisano periods. In this way, for  $k = 1$  the classical Fibonacci sequence is obtained whose Pisano period sequence results

$$\{0, 1, 1, 2, 3, 5, 8, 13, 4, 0, 4, 4, 8, 12, 3, 15, 1, 16, 0, 16, 16, 15, 14, 12, 9, 4, 13, 0, 13, 13, 9, 5, 14, 2, 16, 1\}$$

and hence  $\pi(17) = 36$ . For  $k = 2$  the Pell sequence is obtained whose Pisano period modulo 17 is  $\{0, 1, 2, 5, 12, 12, 2, 16, 0, 16, 15, 12, 5, 5, 15, 1\}$  and, therefore  $\pi_2(17) = 16$ . We may continue in this way for  $k = 3, \dots, 8$  obtaining the period  $36, 16, 16, 12, 36, 36, 8, 68, 68, 8, 36, 36, 12, 16, 16, 36, 2$ .

Next, from the  $k$ -Fibonacci sequences, the periods of the first periodic sequences of Pisano period-lengths are shown. These values have been obtained by varying the value of parameter  $k$  of the  $k$ -Fibonacci sequence.

**4. Pisano periods modulo  $F_{k,n}$**

In this Section, sequences obtained from the remainders of the  $k$ -Fibonacci sequences modulo a  $k$ -Fibonacci number are studied. This Section is straightforwardly deduced from the previous Section. Let us consider the remainders of the first two chains of the sequence  $\{F_{k,m} \bmod F_{k,r}\}$ :

$$0, F_{k,1}, F_{k,2}, \dots, F_{k,r-3}, F_{k,r-2}, F_{k,r-1}, 0, a_1, a_2, a_3, \dots, a_{F_{k,r-1}}. \tag{2}$$

We have then the following result.

**Theorem 17.** In the sequence given in (2)  $a_{2i+1} = F_{k,r-(2i+1)}$  and  $a_{2i} = F_{k,r} - F_{k,r-2i}$ , for  $i = 0, 1, 2, 3, \dots$

**Proof.** It can be proved by induction. Here we show that the result is verified by the first values:

$$\begin{aligned} a_1 &= \text{remainder} \left( \frac{F_{k,r+1}}{F_{k,r}} \right) = \text{remainder} \left( \frac{kF_{k,r} + F_{k,r-1}}{F_{k,r}} \right) = \text{remainder} \left( \frac{F_{k,r-1}}{F_{k,r}} \right) = F_{k,r-1}, \\ a_2 &= \text{remainder} \left( \frac{F_{k,r+2}}{F_{k,r}} \right) = \text{remainder} \left( \frac{kF_{k,r+1} + F_{k,r}}{F_{k,r}} \right) = \text{remainder} \left( \frac{kF_{k,r+1}}{F_{k,r}} \right) = kF_{k,r-1} = F_{k,r} - F_{k,r-2}, \\ a_3 &= \text{remainder} \left( \frac{F_{k,r+3}}{F_{k,r}} \right) = \text{remainder} \left( \frac{kF_{k,r+2} + F_{k,r}}{F_{k,r+1}} \right) = \text{remainder} \left( \frac{kF_{k,r} - kF_{k,r-2} + F_{k,r-1}}{F_{k,r}} \right) \\ &= \text{remainder} \left( \frac{kF_{k,r} + F_{k,r-3}}{F_{k,r}} \right) = kF_{k,r-3}, \\ a_4 &= \text{remainder} \left( \frac{F_{k,r+4}}{F_{k,r}} \right) = \text{remainder} \left( \frac{kF_{k,r+3} + F_{k,r+2}}{F_{k,r+1}} \right) = kF_{k,r-3} + F_{k,r} - F_{k,r-2} = F_{k,r} - F_{k,r-4}. \end{aligned}$$

And so on.  $\square$

As a consequence, following result is obtained.

**Theorem 18.** For every integer  $k \geq 1$ , the number of chains of Pisano periods for the values  $F_{k,r}$ ,  $r = 1, 2, 3, 4, \dots$  is  $\{1, 1, 4, 2, 4, 2, 4, 2, \dots\}$  and where each chain has  $r$  terms.

**Proof.** For  $r = 1$ ,  $F_{k,1} = 1$  and trivially  $\{F_{k,1} \bmod F_{k,1}\} = \{0, 0, 0, \dots\}$ , so there is only one chain in the period  $\{0\}$ .

Now, since  $F_{k,0} = 0$ ,  $F_{k,1} = 1$  and  $F_{k,2} = k$ , the sequence of remainders is  $\{F_{k,n} \bmod F_{k,2}\} = \{0, 1, 0, 1, 0, \dots\}$  by the expression of the  $k$ -Fibonacci polynomials given in Section 1, so the period has only one chain  $\{0, 1\}$ .

Finally, if  $r$  is odd, then  $a_{r-1} = F_{k,r} - F_{k,1} = F_{k,r} - 1$ , so it will be followed by a second chain obtained in the same way, and a fourth chain in which the last term is 1. From this point on, the remainder sequence repeats and so the Pisano period has four chains of  $F_{k,r}$  elements and, as a consequence,  $\pi(F_{k,r}) = 4r$ . On the other hand, if  $r$  is even, then  $a_{r-1} = 1$  and from this point the remainder sequence repeats. So  $\pi(F_{k,r}) = 2r$ .  $\square$

**Corollary 19.** The length of the chains of a  $k$ -Fibonacci sequence modulo the successive  $k$ -Fibonacci numbers form an arithmetic progression. If  $r$  is even greater than 2, then the distance of the progression is 4:  $\{2(2n)\}_{n \geq 2} = \{4n\}_{n \geq 2} = \{8, 12, 16, \dots\}$ . If  $r$  is odd and greater than 1, the distance is 8:  $\{4(2n + 1)\}_{n \geq 1} = \{12, 20, 28, \dots\}$  [25].

For the classical Fibonacci sequence, the lengths of the successive chains are  $\{8, 12, 16, \dots\}$ , while for every odd  $r \geq 3$  the lengths are  $\{12, 20, 28, \dots\}$ . This is because classical Fibonacci is the only  $k$ -Fibonacci sequence in which there are two equal elements  $F_1 = F_2 = 1$ , and so, the sequence of number of chains in the successive Pisano periods is  $\{1, 1, 1, 2, 4, 2, 4, \dots\}$ , and, therefore, the sequence of the lengths of Pisano periods is  $\{1, 1, 3, 8, 20, 12, 28, \dots\}$ .

Table 3  
Pisano period-lengths for  $k$ -Fibonacci sequences

$m \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	<b>1</b>																			
2	<b>3</b>	<b>2</b>																		
3	8	8	<b>2</b>																	
4	6	<b>4</b>	6	<b>2</b>																
5	20	12	12	20	<b>2</b>															
6	24	8	<b>6</b>	8	24	<b>2</b>														
7	16	6	16	16	6	16	<b>2</b>													
8	12	8	12	<b>4</b>	12	8	12	<b>2</b>												
9	24	24	6	8	8	6	24	24	<b>2</b>											
10	60	12	12	20	<b>6</b>	20	12	12	60	<b>2</b>										
11	10	12	8	10	24	24	10	8	12	10	<b>2</b>									
12	24	8	6	8	24	<b>4</b>	24	8	6	8	24	<b>2</b>								
13	28	28	52	28	12	6	6	12	28	52	28	28	<b>2</b>							
14	48	6	48	16	6	16	<b>6</b>	16	6	16	48	6	48	<b>2</b>						
15	40	24	12	40	8	20	24	24	20	8	40	12	24	40	<b>2</b>					
16	24	16	24	8	24	16	24	<b>4</b>	24	16	24	8	24	16	24	<b>2</b>				
17	36	16	16	12	36	36	8	68	68	8	36	36	12	16	16	36	<b>2</b>			
18	24	24	6	8	24	6	24	24	<b>6</b>	24	24	6	24	8	6	24	24	<b>2</b>		
19	18	40	40	6	40	8	40	18	18	18	18	40	8	40	6	40	40	18	<b>2</b>	
20	60	12	12	20	6	20	12	12	60	<b>4</b>	60	12	12	20	6	20	12	12	20	<b>2</b>

Inside the triangle numbers in bold form an axe of symmetry, without the last number in each row.

Table 4  
Pisano periods and their lengths of the 3-Fibonacci sequence

$r$	$F_{3,r}$	$\{F_{3,n}\} \text{ mod } F_{3,r}$	$\pi(r)$
<b>2</b>	3	0,1	$1 \times 2$
<b>3</b>	10	0,1,3,0,3,9,0,9,7,0,7,1	$4 \times 3$
<b>4</b>	33	0,1,3,10,0,10,30,1	$2 \times 4$
<b>5</b>	109	0,1,3,10,33,0,33,99,3,108,0,108,106,99,76,0,76,10,106,1	$4 \times 5$
<b>6</b>	360	0,1,3,10,33,109,0,109,327,10,357,1	$2 \times 6$
<b>7</b>	1189	0,1,3,10,33,109,360,0,360,1080,33,1179,3,1188,0,1188, 1186,1179,1156,1080,829,0,829,109,1156,10,1186,1	$4 \times 7$
<b>8</b>	3297	0,1,3,10,33,109,360,1189,0,1189,3567,109,3894,10,3924,1	$2 \times 8$

**Example 20.** Let us consider the remainders from the 3-Fibonacci sequence:  $\{F_{3,n}\}_{n \in \mathbb{N}} = \{0, 1, 3, 10, 33, 109, 360, 1189, \dots\}$  modulo  $F_{3,n}$  for  $n = 2, 3, 4, \dots$ . The successive Pisano periods and their lengths are shown in Table 4.

Note then, that the sequence of the lengths of Pisano periods from the 3-Fibonacci sequence modulo  $F_{3,r}$ , for  $r = 1, 2, 3, \dots$  is  $\{1, 2, 12, 8, 20, 12, 28, 16, \dots\}$ . Since the third element, the sequence of odd terms is an arithmetic progression with distance 8:  $4\{3, 5, 7, 9, \dots\}$ , while the sequence of even terms is an arithmetic sequence with distance 4:  $4\{2, 3, 4, 5, \dots\}$ .

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