Conditional convergence of numerical series

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One of the most astonishing properties when studying numerical series is that the sum is not commutative, that is the sum may change when the order of its elements is altered. In this note an example is given of such a series. A well-known mathematical proof is given and a MATLAB© program used for different rearrangements of the series converging to different numbers. It is noted that the code follows the mathematical idea of the proof of the property. Developing programs showing the main ideas in the proofs of the theorems may be a good tool in the understanding of such theorems and proofs.

1. Introduction

The sum of an infinite series is very closely connected to the limit of an infinite sequence. In fact, an infinite series is usually defined as the limit of the partial sums of a given sequence $\{a_n\}_{n=1}^{\infty}$. That is

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$$

where

$$S_N = \sum_{n=1}^{N} a_n$$

Finite sums have the property that the sum remains unchanged when the order of the terms is changed, that is

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} b_n$$

for every permutation of $\{1, 2, \ldots, N\}$,φ, such that $b_n = a_{\varphi(n)}$. When we sum infinite series their sums depend on the order of the terms. A change of the order in an infinite series means that a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ is transformed by a rearrangement into a series $\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots$ if every term a_n in the first series appears exactly once in the second and conversely. That is, 'there is a bijection from N to N, φ, such that $b_n = a_{\varphi(n)}$ for every $n \in N$'.

Thus, the only requirement is that every element of the original series appears, only once, somewhere in the new series. If some of the terms are moved to later positions in the series, other terms must be moved to earlier positions. For example the series

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{4}+\dots \qquad (1)$$

is a rearrangement of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}+\dots \qquad (2)$$

However, the sum of the rearrangement (1) is:

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{4}+\dots=\tfrac{3}{2}\ln 2 \qquad (3)$$

while series (2) sums to ln 2 (see for example [1, 2]). In order to see (3), following [3] we can write:

$$\ln 2 = 1\ -\tfrac{1}{2}\ +\tfrac{1}{3}\ -\tfrac{1}{4}\ +\tfrac{1}{5}\ -\tfrac{1}{6}\ +\tfrac{1}{7}\ -\tfrac{1}{8}\ +\ \dots$$

$$\tfrac{1}{2}\ln 2 = \qquad \tfrac{1}{2} \qquad -\tfrac{1}{4} \qquad +\tfrac{1}{6} \qquad -\tfrac{1}{8}\dots$$

Thus, adding both sides of the previous equalities:

$$\tfrac{3}{2}\ln 2 = 1\ +\tfrac{1}{3}\ -\tfrac{1}{2}\ +\tfrac{1}{5}\ +\tfrac{1}{7}\ -\tfrac{1}{4}+\ \dots \qquad (4)$$

However, if a series is the limit of the sequence of the partial sums of a sequence, one may wonder if it is valid to manipulate two infinite series as in the example given, in which the order of the terms is taken in two different ways.

In this note we go over a classic theorem on the importance of the order of terms in infinite sums, and consider the convergence to any number, the divergence or the oscillation of the suitable rearrangement of a series which is conditionally convergent but not absolutely convergent. As an illustration of the theorem we conduct experiments with the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}+\dots\ = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

2. The main theorem

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series but not absolutely convergent. Let $\sum_{n=1}^{\infty} p_n$ be the series formed by the positive terms of the series and $\sum_{n=1}^{\infty} q_n$ the series of the negative terms. Then, both these series are divergent. A rearrangement of the initial series, $\sum_{n=1}^{\infty} b_n$, also exists, such that $\sum_{n=1}^{\infty} b_n$ is divergent, oscillating or convergent to any predetermined number.

Proof. We follow [2]. The proof comprises four steps:

Step 1. The general term of the series, a_n tends to zero when n increases:
By writing

$$a_N = \sum_{n=1}^{N} a_n - \sum_{n=1}^{N-1} a_n = S_N - S_{N-1} \text{ since}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

is convergent, in particular $\{S_N\}_{N=1}^{\infty}$ is a Cauchy sequence, i.e.: for a given positive number ε, it is possible to choose an index $K = K(\varepsilon)$, in such a way that the expression $|S_N - S_M|$ is less than ε, provided only that $N > K$ and $M > K$. In particular $|a_N| = |S_N - S_{N-1}| < \varepsilon$ if $N - 1 > K$, so that $|a_N| \to 0$ when N tends to infinity. Therefore a_n, p_n, and q_n tend to zero when n tends to infinity.

Step 2. The series of positive terms and the series of negative terms are both divergent

Now, since $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, the series $\sum_{n=1}^{\infty} |a_n|$ is divergent, that is $\sum_{n=1}^{\infty} |a_n|$ increases without bound. Let us write

$$\sum_{n=1}^{N} |a_n| = \sum_{n=1}^{N_1} p_n - \sum_{n=1}^{N_2} q_n$$

.

As N increases, N_1 and N_2 increase as well, and the limit of the left-hand side must therefore be equal to the difference of the two sums on the right. Since the series $\sum_{n=1}^{\infty} a_n$ does not converge absolutely, but does converge conditionally, then the series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ must both diverge. Thus, if both were convergent, the series would be absolutely convergent, and if only one of them were divergent the series $\sum_{n=1}^{\infty} a_n$ would be divergent, contrary to the hypothesis.

Step 3. A rearrangement convergent to any predetermined number exists

Now let M be a fixed number. We can rearrange the terms of the series in such a way that the new series $\sum_{n=1}^{\infty} b_n$ converges to M. Suppose that M is positive (this is not a restriction to the proof because in other cases we proceed in a similar way). We then add the first n_1 positive terms, so that the sum $\sum_{n=1}^{n_1} p_n$ is just greater than or equal to M. Since $\sum_{n=1}^{n_1} p_n$ increases with n_1 without bound, it is always possible to make the partial sum greater than or equal to M. We now add just enough negative terms $\sum_{n=1}^{m_1} q_n$ to ensure that the sum $\sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n$ is less than M. This is also possible as follows from the divergence of the series $\sum_{n=1}^{\infty} q_n$.

We now add just enough positive terms $\sum_{n=n_1}^{n_2} p_n$, to make the partial sum again greater than or equal to M.

The values of the sums thus obtained will oscillate around the number M, and when the process is carried far enough the oscillation will take place between arbitrarily narrow bounds; since the terms p_n, and q_n tend to zero when n is sufficiently large, the length of the interval in which the oscillation takes place will also tend to zero. The theorem is thus demonstrated.

Step 4. Divergent and oscillating cases

The same idea as before can be applied to obtain a rearrangement of the initial series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, such that $\sum_{n=1}^{\infty} b_n$ is divergent or oscillating. The key lies in the fact that the positive terms series $\sum_{n=1}^{\infty} p_n$ and negative terms series $\sum_{n=1}^{\infty} q_n$ are both divergent, so we can build up a rearrangement such that $\sum_{n=1}^{n_1} p_n$ is greater than or equal to M, with M being a fixed positive number and

$$\sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n$$

slightly less than M and

$$\sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n + \sum_{n=n_1+1}^{n_2} p_n$$

greater than or equal to $2M$, and so on. In this way the new series is divergent.

Finally, in order to build up an oscillating series it is enough to show that we can obtain partial sums such that, for example

$$\sum_{n=1}^{n_1} p_n > M \qquad \sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n < N \qquad \sum_{n=1}^{n_1} p_n + \sum_{n=1}^{m_1} q_n + \sum_{n=n_1+1}^{n_2} p_n > M$$

and so on. Therefore some partial sums tend to M, and others tend to N. In this way the final rearrangement is neither convergent nor divergent, that is, it is oscillating.

3. An example

In this section we collect some examples regarding different arrangements of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \dots$$

converging to different values. A MATLAB© program was used to produce all these examples.

First, table 1 shows the partial sums

$$\sum_{n=1}^{113} p_n, \sum_{n=1}^{113} p_n + q_1, \sum_{n=1}^{113} p_n + q_1 + \sum_{n=114}^{306} p_n$$

and so on. Table 1 also shows in the first column the number of terms added. Note how the values of the corresponding partial sums of the arrangement get closer to the number 3, lower in the second column, higher in the third one.

Table 1.

Value to be merged: 3 Number of samples	Maximum number of samples Partial sum (by defect)	1000 Partial sum (by excess)
113		3.00328705881207
$(113+1)=114$	2.50328705881207	
$(114+193)=307$		3.00023219266160
$(307+1)=308$	2.75023219266160	
$(308+199)=507$		3.00042246943641
$(507+1)=508$	2.83375580276975	
$(508+201)=709$		3.00114740639264
$(709+1)=710$	2.87614740639264	
$(710+199)=909$		3.00023715645140
$(909+1)=910$	2.90023715645140	

Table 2.

Value to be merged: 4 Number of samples	Maximum number of Samples: Partial sum (by defect)	10 000 Partial sum (by excess)
837		4.00069059163886
$(837+1)=838$	3.50069059163886	
$(838+1,437)=2,275$		4.00026883644081
$(2,275+1)=2,276$	3.75026883644081	
$(2,276+1,475)=3,751$		4.00020358371341
$(3,751+1)=3,752$	3.83353691704674	
$(3,752+1,483)=5,235$		4.00017132261859
$(5,235+1)=5,236$	3.87517132261859	
$(5,236+1,485)=6,721$		4.00008526205086
$(6,721+1)=6,722$	3.90008526205086	
$(6,722+1,487)=8,209$		4.00006885852364
$(8,209+1)=8,210$	3.91673552519031	
$(8,210+1,487)=9,697$		4.00001890244943
$(9,697+1)=9,698$	3.92859033102086	

Table 3.

Value to be merged −3 Number of samples	Maximum number of Samples Partial sum (by defect)	10 000 Partial sum (by excess)
454	−3.00218335417278	
$(454+1)=455$		−2.00218335417278
$(455+2,893)=3,348$	−3.00024278599789	
$(3,348+1)=3,349$		−2.66690945266456
$(3,349+2,901)=6,250$	−3.00009241949378	
$(6,250+1)=6,251$		−2.80009241949378
$(6,251+3,476)=9,726$	−3.00003131115295	
$(9,726+1)=9,727$		−2.85717416829581

Analogously table 2 shows the partial sums of the rearrangement tending to 4. 10 000 terms have been added in this example.

Tables 3 and 4 respectively show the evolution of the partial sums for the limits −3 and 5. Note that when the limit is further away from $\ln 2$ (the sum of the initial rearrangement) more terms have to be added to obtain the predetermined number.

4. Conclusion

A program written in MATLAB© is used to demonstrate the conditional convergence of series. The program is available from the author on request.

Table 4.

Value to be merged 5 Number of samples	Maximum number of Samples Partial sum (by defect)	125 000 Partial sum (by excess)
6,183		5.00004172004347
$(6,183+1)=6,184$	4.50004172004347	
$(6,184+10,625)=16,809$		5.00004603622696
$(16,809+1)=16,810$	4.75004603622696	
$(16,810+10,903)=27,713$		5.00002791266854
$(27,713+1)=27,714$	4.83336124600187	
$(27,714+10,961)=38,675$		5.00000202192147
$(38,675+1)=38,676$	4.87500202192147	
$(38,676+10,985)=49,661$		5.00001242215457
$(49,661+1)=49,662$	4.90001242215457	
$(49,662+10,993)=60,655$		5.00000167496475
$(60,655+1)=60,656$	4.91666834163142	
$(60,656+11,001)=71,657$		5.00001146319518
$(71,657+1)=71,658$	4.92858289176661	
$(71,658+17,003)=82,661$		5.00001049295067
$(82,661+1)=82,662$	4.93751049295067	
$(82,662+11,005)=93,667$		5.00000879852260
$(93,667+1)=93,668$	4.94445324296704	
$(93,668+11,005)=104,673$		5.00000031448531
$(104,673+1)=104,674$	4.95000031448531	
$(104,674+11,009)=115,683$		5.00000610322352
$(115,683+1)=115,684$	4.95455155776897	

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A magic decomposition

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A lively example to use in a first course in linear algebra to clarify vector space notions is the space of square matrices of fixed order with its subspaces of affine, coaffine, doubly affine, and magic squares. In this note, the projection theorem is illustrated by explicitly constructing the orthogonal projections (in closed forms) of any matrix **U** onto these subspaces. The results follow directly from a canonical decomposition of **U**.