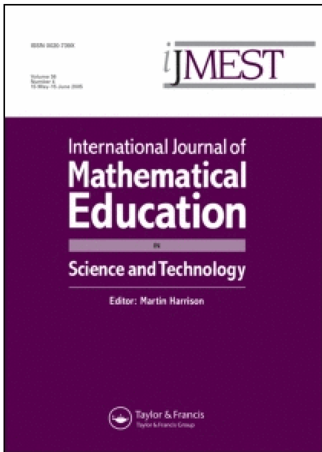

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Identities for generalized Fibonacci numbers: a combinatorial approach

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This note shows a combinatorial approach to some identities for generalized Fibonacci numbers. While it is a straightforward task to prove these identities with induction, and also by arithmetical manipulations such as rearrangements, the approach used here is quite simple to follow and eventually reduces the proof to a counting problem.

Keywords: generalized Fibonacci numbers; combinatorial proof; non-inductive methods

1. Introduction

One of the simplest and more studied integer sequences is the Fibonacci sequence [1,2]: $\{F_n\}_{n=0}^{\infty} = \{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ \ldots\}$ wherein each term is the sum of the two preceding terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. Fibonacci numbers arise in the solution of many combinatorial problems. They count the number of binary sequences with no consecutive zeros, the number of sequences of 1's and 2's which sum to a given number, the number of independent sets of a path graph, etc. These interpretations have been used to provide combinatorial proofs of many interesting Fibonacci, and also Lucas and binomial identities [3–5].

Fibonacci numbers have been generalized in many ways. Here, we use the k-Fibonacci numbers as studied in [6,7], which depend only on one integer variable k. For any integer number $k \geq 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by: $F_{k,0} = 0$, $F_{k,1} = 1$, and $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$. Particular cases of k-Fibonacci numbers are:

- If $k = 1$, the classical Fibonacci sequence is obtained: $\{F_n\}_{n\in\mathbf{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$.
- If $k = 2$, the Pell sequence appears: $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$: $\{P_n\}_{n\in\mathbf{N}} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}$.
- If $k = 3$, the following sequence appears: $F_{3,0} = 0$, $F_{3,1} = 1$, and $F_{3,n+1} = 3F_{3,n} + F_{3,n-1}$ for $n \geq 1$: $\{F3, n\}_{n\in\mathbf{N}} = 0, 1, 3, 10, 33, 109, \ldots\}$.

The first 10 k-Fibonacci sequences listed in *The On-Line Encyclopedia of Integer Sequences* [8] with their reference numbers are given in Table 1.

k-Fibonacci numbers satisfy numerous relationships, as the classical Fibonacci numbers. Many of these identities for the classical Fibonacci numbers are documented in [2], where they are proved by algebraic means. Some of these formulas are combinatorially proved in [5].

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Our goal is to provide combinatorial proofs of some identities [9] for the generalized k-Fibonacci numbers. We show that in the context of "colour-square tilings", these identities follow naturally as the tilings are counted.

1.1. *Combinatorial interpretation*

F_{n+1} counts the number of ways to tile a $1 \times n$ rectangle (called an *n-board* consisting of *cells* labelled $1, 2, \ldots, n$) with 1×1 *squares* and 1×2 *dominoes*. For combinatorial convenience it is defined $f_n = F_{n+1}$ [5].

In the same way, for the k-Fibonacci numbers we shall obtain an analogous combinatorial interpretation. Define a *colour-square* tiling to be a tiling of an n-board by *colour-squares* and non-colour (or black) dominoes. If there are k different colours to choose for the squares the tilings generated in this way for an n-board are precisely $f_{k,n} = F_{k,n+1}$. For example, the tiling in Figure 1 has two black dominoes followed by a colour string of length 4, and so on.

By conditioning on whether the first tile is a square or a domino, we obtain the identity $f_n = kf_{n-1} + f_{n-2}$. In addition, for convenience, we consider $f_0 = 1$ the number of tilings for the empty 0-board.

Two of the three identities proved in [9] by non-inductive proofs are below. We will prove them by combinatorial arguments, which are also non-inductive.

- *Identity* 1

$$\sum_{j=1}^{n} F_{k,j} = \frac{F_{k,n} + F_{k,n+1} - 1}{k}, \qquad \text{that is} \qquad \sum_{j=0}^{n-1} f_{k,j} = \frac{f_{k,n} + f_{k,n+1} - 1}{k}$$

- *Identity* 2

$$\sum_{j=1}^{n} (F_{k,j})^2 = \frac{F_{k,n}F_{k,n+1}}{k}, \qquad \text{that is} \qquad \sum_{j=0}^{n-1} (f_{k,j})^2 = \frac{f_{k,n}f_{k,n+1}}{k}$$

2. Combinational proofs

We now show how to establish these results by combinatorial arguments. The general idea behind these non-inductive proofs is to count the elements of some set by two ways, each corresponding to one of the sides of the identity we are proving [5].

Table 1. The first 10 k-fibonacci series [8].

$\{F_{1,n}\}$	*A*000045	$\{F_{6,n}\}$	*A*005668
$\{F_{2,n}\}$	*A*000129	$\{F_{7,n}\}$	*A*054413
$\{F_{3,n}\}$	*A*006190	$\{F_{8,n}\}$	*A*041025
$\{F_{4,n}\}$	*A*001076	$\{F_{9,n}\}$	*A*099371
$\{F_{5,n}\}$	*A*052918	$\{F_{10,n}\}$	*A*041041

Figure 1. Example of a 4-colour 12-board with three dominoes.

Identity 1 follows as a consequence of the next two identities, which cover the cases in which in the left-hand side (LHS) appear terms of the form $f_{k,2j-1}$ and $f_{k,2j}$, respectively:

- *Identity* 1(a)

$$\sum_{j=1}^{n} f_{k,2j-1} = \frac{f_{k,2n} - 1}{k}$$

Proof: Let us write this identity as $\sum_{j=1}^{n} kf_{k,2j-1} = f_{k,2n} - 1$. Then the (RHS) is the number of k-colour square tilings of an $2n$-board containing at least 1 colour square, that is $f_{2n} - 1$ since we have to discount the case of all domino tiling. On the other hand, the (LHS) is obtained by conditioning on the position of the last k-colour square. See Figure 2. □

- *Identity* 1(b)

$$\sum_{j=0}^{n} f_{k,2j} = \frac{f_{k,2n+1}}{k}$$

Proof: Let us write this identity as $\sum_{j=0}^{n} kf_{k,2j} = f_{k,2n+1}$. Then the (RHS) is the number of colour-square tilings of a $(2n+1)$-board, while the (LHS) is obtained by conditioning on the position of the last k-colour square. See Figure 3. □

- *Identity* 2

$$\sum_{j=0}^{n-1} (f_{k,j})^2 = \frac{f_{k,n} f_{k,n+1}}{k}$$

Proof: Identities involving squares of k-Fibonacci numbers suggest investigating pairs of k-colour square tilings.

Let us write this identity as $\sum_{j=0}^{n-1} k\,(f_{k,j})^2 = f_{k,n} f_{k,n+1}$. Then, the (RHS) counts ordered pairs (A, B) of k-colour square (n & $n+1$)-tilings, where A or B contains at least one square. Following [5] we can interpret the (LHS) by defining the parameter m_X to be the first cell of the k-colour square tiling X covered by a square. If X is all dominoes, we set m_X equal to infinity. Since, in our case, at least one square exists in (A, B), the minimum of m_A and m_B must be finite and odd. Let $m = \min\{m_A + 1, m_B\}$. When m is odd, A and B have dominoes covering cells 1 through $m-1$ and B has a square covering cell m.

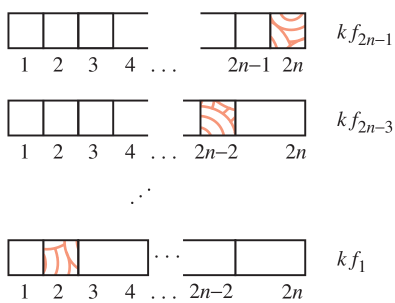

Figure 2. A $2n$-board with at least one k-colour square. The last k-color square is followed by dominoes.

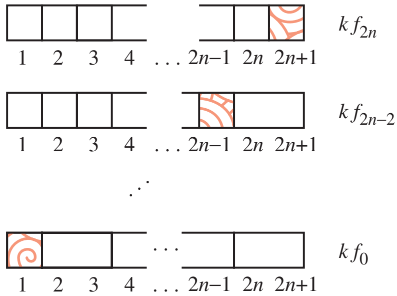

Figure 3. A $(2n+1)$-board has at least one k-colour square. The last k-colour square is followed by dominoes.

Hence, the number of k-colour square pairs (A, B) with odd m is $kf_{k,n-m+1}f_{k,n-m+1}$. If m is even, B has dominoes covering cells 1 through m and A has dominoes covering cells 1 through $m-2$ with a k-colour square covering cell $m-1$. Hence, the number of k-colour square pairs (A, B) with m even is also $kf_{k,n+1-m}f_{k,n+1-m}$. Setting $j = n+1-m$ gives the desired identity. □

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