International Journal of Mathematical Education in Science and Technology, Vol. 39, No. 6, 15 September 2008, 785–792

Combinatorial proofs of Honsberger-type identities

A. Plaza* and S. Falcón

Department of Mathematics, University of Las Palmas de Gran Canaria, 35017-Las Palmas de Gran Canaria, Spain

(*Received 8 October 2007*)

In this article, we consider some generalizations of Fibonacci numbers. We consider k-Fibonacci numbers (that follow the recurrence rule $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$), the (k, ℓ)-Fibonacci numbers (that follow the recurrence rule $F_{k,n+2} = kF_{k,n+1} + \ell F_{k,n}$), and the Fibonacci p-step numbers ($F_p(n) = F_p(n-1) + F_p(n-2) + \cdots + F_p(n-p)$, with $n > p+1$, and $p > 2$). Then we provide combinatorial interpretations of these numbers as square and domino tilings of n-boards, and by easy combinatorial arguments Honsberger identities for these Fibonacci-like numbers are given. While it is a straightforward task to prove these identities with induction, and also by arithmetical manipulations such as rearrangements, the approach used here is quite simple to follow and eventually reduces the proof to a counting problem.

Keywords: generalized Fibonacci numbers; combinatorial proof; Honsberger identities

1. Introduction

One of the simplest and more studied integer sequences is the Fibonacci sequence [1–4]: $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, \ldots\}$ wherein each term is the sum of the two preceding terms, beginning with the values $F_0 = 0$, and $F_1 = 1$. Fibonacci numbers arise in the solution of many combinatorial problems. They count the number of binary sequences with no consecutive zeros, the number of sequences of 1's and 2's which sum to a given number, the number of independent sets of a path graph, etc. These interpretations have been used to provide combinatorial proofs of many interesting Fibonacci, and also Lucas and binomial identities [5–8].

Fibonacci numbers have been generalized in many ways. Here we use the k-Fibonacci numbers as studied in [9,10], which depend only on one integer variable k. For any integer number $k \geq 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by: $F_{k,0} = 0, F_{k,1} = 1$ and $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$. Particular cases of k-Fibonacci numbers are:

- If $k = 1$, the classical Fibonacci sequence is obtained: $\{F_n\}_{n\in\mathbf{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$.
- If $k = 2$, the Pell sequence appears: $\{P_n\}_{n\in\mathbf{N}} = \{0, 1, 2, 5, 12, 29, 70, \ldots\}$.
- If $k = 3$, the following sequence appears: $\{F_{3,n}\}_{n\in\mathbf{N}} = \{0, 1, 3, 10, 33, 109, \ldots\}$.

*Corresponding author. Email: aplaza@dmat.ulpgc.es

ISSN 0020–739X print/ISSN 1464–5211 online

DOI: 10.1080/00207390801986916
http://www.informaworld.com

Table 1. The first 12 k-Fibonacci sequences listed in [11].

$\{F_{1,n}\}$	$A000045$	$\{F_{7,n}\}$	$A054413 \cup \{0\}$
$\{F_{2,n}\}$	$A000129$	$\{F_{8,n}\}$	$A041025 \cup \{0\}$
$\{F_{3,n}\}$	$A006190$	$\{F_{9,n}\}$	$A099371$
$\{F_{4,n}\}$	$A001076$	$\{F_{10,n}\}$	$A041041 \cup \{0\}$
$\{F_{5,n}\}$	$A052918 \cup \{0\}$	$\{F_{11,n}\}$	$A049666$
$\{F_{6,n}\}$	$A005668$	$\{F_{12,n}\}$	$A041061 \cup \{0\}$

Table 2. Examples of (k, ℓ)-Fibonacci sequences listed in *OEIS*.

$k \backslash \ell$	2	3	4	5
1	$A001055$	$A006130$	$A006131$	$A015440$
2	$A002605$	$A015518$	$A063727$	$A002532$
3	$A007482$	$A030195$	$A015521$	$A015523$
4	$A090017$	$A015530$	$A057087$	$A015531$
5	$A015535$	$A015536$	$A015537$	$A057088$

It is worthy to be noted that only the first 11 k-Fibonacci sequences are listed in *The On-Line Encyclopedia of Integer Sequences* [11], from now on *OEIS*, with the numbers given in Table 1.

k-Fibonacci numbers satisfy numerous relationships, as the classical Fibonacci numbers. Many of these identities for the classical Fibonacci numbers are documented in [4], where they are proved by algebraic means. Some of these formulas are combinatorially proved in [6,7].

Fibonacci numbers may also be generalized considering the 2-parameter recurrence relation: $G_{n+1} = kG_n + \ell G_{n-1}$ for $n \geq 1$, with initial conditions $G_0 = 0$, $G_1 = 1$. We use letter G for these numbers that will be called here (k, ℓ)-Fibonacci numbers. Table 2 shows some examples of these sequences as listed in *OEIS*.

Finally, here we will also consider the Fibonacci p-step numbers also known as higher-order Fibonacci numbers [12]. Feinberg extended the summation property $F_n = F_{n-1} + F_{n-2}$ of the Fibonacci sequence to $F_n = F_{n-1} + F_{n-2} + F_{n-3}$ [13]. The new numbers were named tribonacci numbers, because now addition of three successive members in the sequence give the next member. In general, the Fibonacci p-step numbers are defined by the recurrence relation $F_p(n) = F_p(n-1) + F_p(n-2) + \cdots + F_p(n-p)$, with $n \geq p$, and $p > 2$. Extending also the initial conditions from the classical Fibonacci, for the Fibonacci p-step numbers are: $F_p(n) = 0$, if $0 \leq n \leq p-2$, $F_p(p-1) = 1$. With these initial conditions, as it can be easily checked, the first non-null numbers in the sequence are $1, 1, 2, 4, \ldots, 2^{p-1}$. Therefore, and also in order to use a combinatorial interpretation of these numbers we will employ here the following initial conditions for the Fibonacci p-step numbers: $F_p(0) = 0$, $F_p(1) = 1, F_p(n) = 2^{n-2}$, for $2 \leq n \leq p-1$ [13,14]. Some of the first Fibonacci p-step numbers along with their references in *OEIS* are shown in Table 3.

Our goal is to provide Honsberger-type identities [2] for the Fibonacci, k-Fibonacci and (k, ℓ)-Fibonacci numbers by combinatorial means. We show that in the context of 'colour-square tilings', these identities follow naturally as the tilings are counted.

Table 3. The first Fibonacci p-step numbers without initial zeros.

p	Sloane	Name	Sequence
3	*A*000073	Tribonacci	$\{1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots\}$
4	*A*000078	Tetranacci	$\{1, 1, 2, 4, 8, 15, 29, 56, 108, \ldots\}$
5	*A*001591	Pentanacci	$\{1, 1, 2, 4, 8, 16, 31, 61, 120, \ldots\}$
6	*A*001592	Hexanacci	$\{1, 1, 2, 4, 8, 16, 32, 63, 125, \ldots\}$
7	*A*122189	Heptanacci	$\{1, 1, 2, 4, 8, 16, 32, 64, 127, \ldots\}$

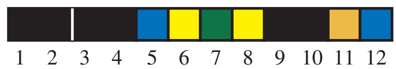

Figure 1. A 4-colour 12-board with three dominoes.

1.1. *Combinatorial interpretation*

F_{n+1} counts the number of ways to tile a $1 \times n$ rectangle (called an *n-board* consisting of *cells* labeled $1, 2, \ldots, n$) with 1×1 *squares* and 1×2 *dominoes*. For combinatorial convenience it is defined $f_n = F_{n+1}$ [6,7].

In the same way, for the k-Fibonacci numbers we shall obtain an analogous combinatorial interpretation. Define a *colour*-square tiling to be a tiling of an n-board by *colour*-squares and non-colour (or black) dominoes. If there are k different colours to choose for the squares, the tilings generated in this way for an n-board are precisely $f_{k,n} = F_{k,n+1}$. From now on, we will write f_n and F_n omitting sub-index k. For example, the tiling in Figure 1 has two black dominoes followed by a colour string of length 4, and so on.

By conditioning on whether the first tile is a square or a domino, we obtain the identity $f_n = kf_{n-1} + f_{n-2}$. In addition, for convenience, we consider $f_0 = 1$ the number of tilings for the empty 0-board.

It should be noted that if k colours are allowed for squares and ℓ colours are permitted for dominoes, and g_n represents the number of ways to tile an n-board with k-colour squares and ℓ-colour dominoes, by conditioning on whether the first tile is a square or a domino, we obtain the identity $g_n = kg_{n-1} + \ell g_{n-2}$. In addition, for convenience, we consider $g_0 = 1$ the number of tilings for the empty 0-board. Note that $g_n = G_{n+1}$, where G_n are the (k, ℓ)-Fibonacci numbers as presented in a previous subsection.

For the Fibonacci p-numbers the combinatorial interpretation follows by permitting squares, dominoes and longer tiles until p-tiles in each n-board. If h_n is the number of tilings obtained in this way, then, as one can be checked easily, $h_n = F_p(n+1)$. As before, we consider $h_0 = 1$.

2. Honsberger identities

From now on we will use, following [7] the concepts of *breakable* tiling and *unbreakable* tiling. It is said that a tiling of an n-board is *breakable* at cell p, if the tiling can be decomposed into two tilings, one covering cells 1 through p and the other covering cells $p+1$ through n. On the other hand, a tiling is said to be *unbreakable* at cell p if a domino occupies cells p and $p+1$ (Figure 2).

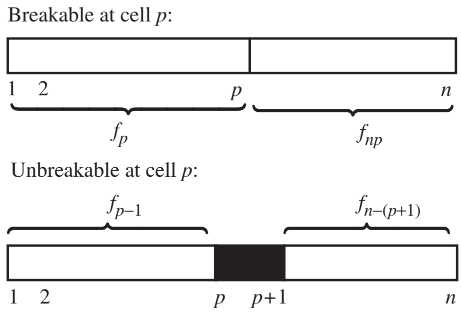

Figure 2. An (n)-board is either breakable or unbreakable at cell p.

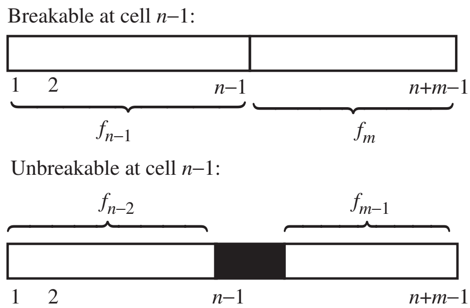

Figure 3. A $(n+m-1)$-board is either unbreakable or breakable at cell $n-1$.

For example, the tiling of Figure 1 is breakable at cells $2, 4, 5, 6, 7, 8, 10, 11$ and 12. Observe that an n-tiling is always breakable at cell n.

Honsberger [2, p. 107] gives the following general relation for classical Fibonacci numbers:

Honsberger Identities for classical Fibonacci numbers:

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

This identity may be written as

$$f_{n+m-1} = f_{n-2}f_{m-1} + f_{n-1}f_m \tag{1}$$

where $f_n = F_{n+1}$ counts the number of n-tilings with squares and black dominoes.

Proof: Note, that Identity (1) is proved very easily considering the two possibilities for cell $n-1$ in an $(n+m-1)$-tiling. Either cell $n-2$ is covered by the beginning of a domino or is not. In the first case, it is said that the tiling is unbreakable at cell $n-1$. Otherwise it is said that the tiling is breakable at cell $n-1$. And hence, Identity (1) is proved (Figure 3).

Notice that the argument is directly applicable to k-Fibonacci numbers since the colour does not apply here. This would not be the case if we consider n-tilings by colour squares and colour dominoes. Then the identity changes accordingly to the number of colours allowed for the dominoes.

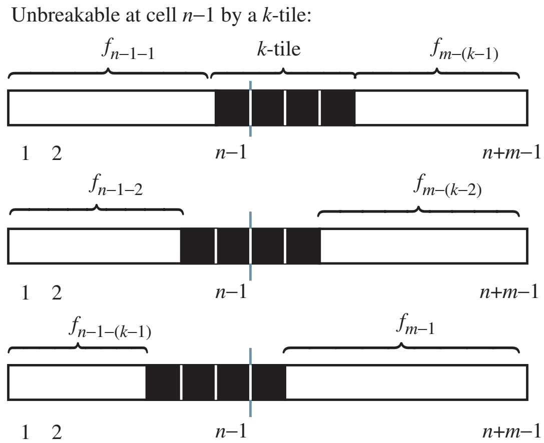

Figure 4. A $(n+m-1)$-board is unbreakable at cell $n-1$ by an (unbreakable) k-tile in $k-1$ positions of the k-tile ($k=4$ here).

Corollary 1: *If now $f_p = F_{k,p+1}$ counts the number of k-colour square p-tilings with (k-colour) squares and (black) dominoes, then Equation (1) applies.*

If g_n denotes the number of k-colour square and ℓ-colour domino n-tilings with (k-colour) squares and (ℓ-colour) dominoes, then the analogous equation to Equation (1) is:

$$g_{n+m-1} = \ell \cdot g_{n-2}g_{m-1} + g_{n-1}g_m \tag{2}$$

If $h_n = F_p(n+1)$ denotes the number of n-tilings where squares, dominoes, etc. until p-tiles are permitted, then the analogous equation to Equation (1) is:

$$h_{n+m-1} = h_{n-1}h_m + \sum_{k=2}^{p}\sum_{j=1}^{k-1} h_{n-1-j}h_{m-(k-j)} \tag{3}$$

Note that for the Fibonacci p-step numbers each $(n+m-1)$-board is unbreakable cell $n-1$ by a k-tile in $k-1$ positions of the k-tile, as is shown in Figure 4 for the value $k=4$.

Now we shall extend the Honsberger identities by considering more segments (breakable or unbreakable) in the corresponding tiling. Next we take three terms in previous formulas.

3. 3-Term Honsberger identities

3-Term Honsberger Identities for classical Fibonacci numbers:

$$f_{n+m+p} = f_n f_m f_p + f_{n-1} f_{m-1} f_p + f_n f_{m-1} f_{p-1} + f_{n-1} f_{m-2} f_{p-1} \tag{4}$$

Proof: The result follows immediately by considering the two disjoint possibilities (breakable or unbreakable) for the cells n and $n+m$ (Figure 5).

If the tiling is breakable at cells n and $n+m$ there are $f_n f_m f_p$ such tilings. In other cases, if the board is breakable at cell n and unbreakable at cell $n+m$ it results in

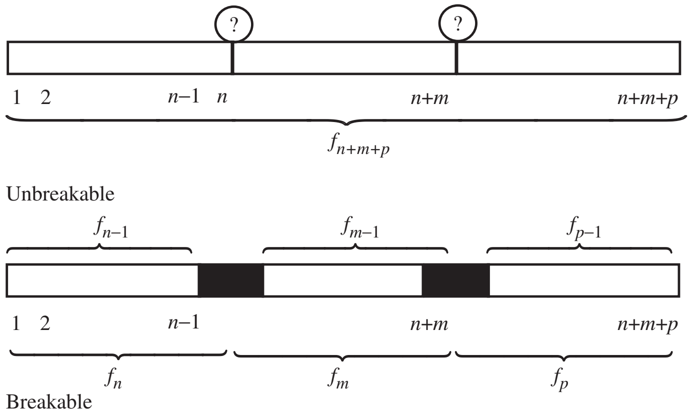

Figure 5. A $(n+m+p)$-board is either unbreakable or breakable at cells n, and $n+m$.

$f_{n-1}f_{m-1}f_p$ tilings. If the board in unbreakable at cell n and breakable at cell $n+m$ it results in $f_nf_{m-1}f_{p-1}$. Finally, if the board is breakable both at cell n and at cell $n+m$ there are $f_{n-1}f_{m-2}f_{p-1}$ such tilings.

Notice that Equation (4) may be written as follows:

$$f_{n+m+p} = \sum_{i,j=0}^{1} f_{n-i}f_{m-(i+j)}f_{p-j} \tag{5}$$

Corollary 3: *In the case $n = m = p$, Equation* (4) *reads as $f_{3n} = f_n^3 + 2f_{n-1}^2 f_n + f_{n-1}^2 f_{n-2}$, or equivalently $F_{k,n}^2 = (F_{k,3n+1} - F_{k,n+1}^3)/(2F_{k,n+1} + F_{k,n-1})$.*

Note that previous formula tells that, in particular, the quotient $(F_{k,3n+1} - F_{k,n+1}^3)/(2F_{k,n+1} + F_{k,n-1})$ is a perfect square. For example, for $k = 3$ and $n = 4$ it is $(F_{3,13} - F_{3,5}^3)/(2F_{3,5} + F_{3,3}) = F_{3,4}^2$, that is $(1,543,321 - 109^3)/(2 \cdot 109 + 10) = 1089 = 33^2$.

Corollary 4: *Now if $f_p = F_{k,p+1}$ counts the number of k-colour square p-tilings with* (*k-colour*) *squares and* (*black*) *dominoes, then Equation* (4) *applies.*

However, if g_p denotes the number of k-colour square and ℓ-colour domino p-tilings with (k-colour) squares and (ℓ-colour) dominoes, then the analogous equation to Equation (1) is:

$$g_{n+m+p} = g_n g_m g_p + \ell g_{n-1}g_{m-1}g_p + \ell g_n g_{m-1}g_{p-1} + \ell^2 g_{n-1}g_{m-2}g_{p-1} \tag{6}$$

4. Generalization

Previous expressions may be extended to 4-term Honsberger or more generally m-terms. Here only the results for k-Fibonacci and (k,ℓ)-Fibonacci numbers are shown. We omit the proofs because they are based on the same considerations as before, taking into account the two possibilities (breakable or unbreakable) between two adjacent segments in the corresponding tiling.

4.1. *4-Term Honsberger identities*

For classical Fibonacci numbers (with $f_p = F_{p+1}$), or k-Fibonacci numbers (with $f_p = F_{k,p+1}$), then:

$$f_{n_1+n_2+n_3+n_4} = \sum_{i_1,\, i_2,\, i_3=0}^{1} f_{n_1-i_1} f_{n_2-(i_1+i_2)} f_{n_3-(i_2+i_3)} f_{n_4-i_3} \tag{7}$$

For (k, ℓ)-Fibonacci numbers (with $g_p = G_{p+1}$), then:

$$g_{n_1+n_2+n_3+n_4} = \sum_{i_1,\, i_2,\, i_3=0}^{1} \ell^{i_1+i_2+i_3} g_{n_1-i_1} g_{n_2-(i_1+i_2)} g_{n_3-(i_2+i_3)} g_{n_4-i_3} \tag{8}$$

4.2. *m-Term Honsberger identities*

For classical Fibonacci numbers (with $f_p = F_{p+1}$), or k-Fibonacci numbers (with $f_p = F_{k,p+1}$), then:

$$f_{n_1+\cdots+n_m} = \sum_{i_1,\ldots,\, i_{m-1}=0}^{1} f_{n_1-i_1} f_{n_2-(i_1+i_2)} \cdots f_{n_m-i_{m-1}} \tag{9}$$

For (k, ℓ)-Fibonacci numbers (with $g_p = G_{p+1}$), then:

$$g_{n_1+\cdots+n_m} = \sum_{i_1,\ldots,\, i_{m-1}=0}^{1} \ell^{i_1+\cdots+i_{m-1}} g_{n_1-i_1} g_{n_2-(i_1+i_2)} \cdots g_{n_m-i_{m-1}} \tag{10}$$

5. Conclusions

The techniques presented in this article are simple and powerful. Counting k-colour square tilings enables us to give visual interpretations to Honsberger-type expressions involving k-Fibonacci numbers, (k, ℓ)-Fibonacci numbers or Fibonacci p-step numbers. Similar arguments are also applicable to other identities.

Acknowledgements

This work has been supported in part by CICYT Project number MTM2005-08441-C02-02 from Ministerio de Educación y Ciencia of Spain.

References

[1] V.E. Hoggat, *Fibonacci and Lucas numbers*, Houghton-Mifflin, Palo Alto, CA, 1969.
[2] R. Honsberger, *A second look at the Fibonacci and lucas numbers*, in *Mathematical Gems III*, Mathematical Association of America, Washington, 1985.
[3] E. Kilic, *The Binet formula, sums and representations of generalized Fibonacci p-numbers*, Eur. J. Comb. (2007) doi:10.1016/j.ejc.2007.03.004.
[4] S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section*, Theory Appl., Ellis Horwood Limited, Chichester, UK, 1989.
[5] A.T. Benjamin and J.J. Quinn, *Phased tilings and generalized Fibonacci identities*, Fibonacci Quart. 38 (2000), pp. 282–288.

[6] A.T. Benjamin and J.J. Quinn, *Fibonacci numbers – exposed more discretely*, Math. Mag. 76 (2003), pp. 182–192.
[7] A.T. Benjamin and J.J. Quinn, *Proofs that Really Count. The Art of Combinatorial Proof*, The Mathematical Association of America, Washington, DC, 2003.
[8] R.C. Bringham et al., *A tiling scheme for the Fibonacci numbers*, J. Recreational Math. 28 (1996–1997), pp. 10–16.
[9] S. Falcón and A. Plaza, *On the Fibonacci k-numbers*, Chaos, Sol. Fract. 32 (2007), pp. 1615–1624.
[10] S. Falcón and A. Plaza, *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos, Sol. Fract. 33 (2007), pp. 38–49.
[11] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences (2006). Available at www.research.att.com/~njas/sequences/
[12] M. Radić, D.A. Morales and O. Araujo, *Higher-order Fibonacci numbers*, J. Math. Chem. 20 (1996), pp. 79–94.
[13] M. Feinberg, *Fibonacci-Tribonacci*, Fibonacci Quart. 1 (1963), pp. 71–74.
[14] E.W. Weisstein, Fibonacci *n*-step number, published electronically. Available at mathworld. wolfram.com/Fibonaccin-StepNumber.html