Function	Zeros	Maxima	Minima	Points of inflexion
$f(x)$	0			0 (a) $\sqrt{3}$ (b)
$f^1(x)$	0 (a)	$\sqrt{3}$ (b)	0 (c)	0.485 028 (d)
$f^2(x)$	0 (a), (c) $\sqrt{3}$ (b)	0.485 028 (d)	2.895 602 1 (e)	0.867 554 2 (f) 4.014 175 2 (g)
$f^3(x)$	0.485 028 (d) 2.895 602 1 (e)	0 (h) 4.014 175 2 (g)	0.867 554 2 (f)	0.305 977 7 (i) 1.208 590 5 (j) 5.117 278 2 (k)
$f^4(x)$	0 (h) 0.867 554 2 (f)	1.208 590 5 (j)	0.305 977 7 (i) 5.117 278 2 (k)	0.548 495 6 (l) 1.529 626 2
(m)				
	4.014 175 2 (g)			6.213 159 2 (n)
$f^5(x)$	0.305 977 7 (i) 1.208 590 5 (j) 5.117 278 2 (k)	0.548 495 6 (l) 6.213 159 2 (n)	0 (o) 1.529 626 2 (m)	to be determined
$f^6(x)$	0 (o) 0.548 4956 (l) 1.529 626 2 (m) 6.213 159 2 (n)	to be determined	to be determined	to be determined

We leave readers to investigate further properties of $f(x)$, and whether there are other functions with such an interesting structure.

Two approximations to π

J. L. BONET,[1] A. PLAZA[2] and M. A. PADRÓN[2]

[1] Isabel de España High School, Las Palmas de Gran Canaria, Spain.
[2] Department of Mathematics. University of Las Palmas de Gran Canaria, Spain.

(*Received 25 February 1998; revised 21 September 1998*)

Two well known approximations of number π are shown with some detail. For this purpose two sequences of approximating regular convex polygons are used. The polygons are inscribed in the limit circle for the defect approximation and circumscribed for the excess approximation. In the later case, the numerical scheme is slightly changed in order to improve the convergence, and to avoid the subtractive cancellation and division errors inherent to the former approximation. This improvement is measured by comparison of the error table in both cases.

1. Introduction

What is π? An approximate answer is that π equals 3.14. Another common approximation is that

$$\pi \cong \frac{22}{7}$$

There even exists an organization, the 1000 Club, whose only membership requirement is a perfect recollection of the first 1000 digits of pi [1]. But these are only rational approximations for an irrational number. Since it is well known that π is an irrational number, the previous answers are clearly false. A better answer is that π is the ratio of the circumference of a circle to its diameter. This, of course, is true, but if you then ask what is the circumference of a circle, you are apt to be told that it is equal to $2\pi r$. Now you have a definition that essentially defines π in terms of itself. A different method of defining π approaches the problem in terms of the limit of a sequence of perimeters of regular polygons inscribed in a circle. See, for example [2, 3].

2. Approximating π by defect

Let P_m be the regular polygon of m sides, inscribed in the unit circle. If A_m is the area of P_m, note first that

$$A_{2m} = \frac{m}{2}\sqrt{2 - 2\sqrt{1 - (2A_m/m)^2}} \tag{1}$$

Let OAB be a semi-triangle centred on the X-axis and within the area A_m, and let OAC be the triangle corresponding to the polygon A_{2m}, as shown in figure 1.

From figure 1, the following equations follow:

$$x^2 + y^2 = 1$$

$$\frac{xy}{2} = areaOAB$$

Eliminating x, the following bi-quadratic equation is obtained:

$$y^4 - y^2 + 4(areaOAB)^2 = 0$$

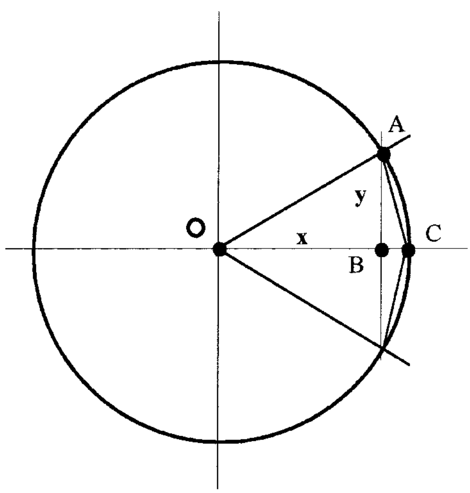

Figure 1.

Noting that there is only one admissible value for y (since $0 < y < 1$); we obtain

$$y = \left[\frac{1 - (1 - 16(areaOAB)^2)^{1/2}}{2}\right]^{1/2}$$

and

$$areaOAC = \frac{1}{2}y = \frac{1}{2}\left[\frac{1 - (1 - 16(areaOAB)^2)^{1/2}}{2}\right]^{1/2}$$

so

$$A_{2m} = 2m \times areaOAC = \frac{2m}{2}\left[\frac{1 - \left(1 - 16(A_m/2m)^2\right)^{1/2}}{2}\right]^{1/2}$$

$$= \frac{m}{2}\left[2 - 2\left(1 - (4A_m/2m)^2\right)^{1/2}\right]^{1/2}$$

and the relation (1) is proved.

So, each term of the $\{A_m\}$ gives an approximation of π. We find:

$$A_4 = 2 \qquad A_8 = 2\sqrt{2} = 2.828\,427 \qquad A_{16} = 4(2 - \sqrt{2})^{1/2} = 3.061\,467$$

$$A_{32} = 8[2 - (2 + \sqrt{2})^{1/2}]^{1/2} = 3.121\,445 \qquad \dots \qquad A_{4096} = 3.141\,592\dots$$

The evolution of the round-off error for the previous scheme is shown in table 1. Note that in this way π can be written as the following limit:

$$\pi = \lim_{m\to\infty} A_{2^m} = \lim_{m\to\infty} 2^{m-2}\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \dots \sqrt{2}}}}_{m-3}}$$

m	π	Error
1	2.0	1.141 592 653 589 79
2	2.828 427 124 746 19	3.131 655 288 43E-01
3	3.061 467 458 920 72	8.012 519 466 907 441E-02
4	3.121 445 152 258 05	2.014 750 133 174 026E-02
5	3.136 548 490 545 94	5.044 163 043 852 468E-03
6	3.140 331 156 954 74	1.261 496 635 053 926E-03
7	3.141 277 250 932 76	3.154 026 570 362 234E-04
8	3.141 513 801 144 15	7.885 244 564 764 804E-05
9	3.141 572 940 367 88	1.971 322 191 041 124E-05
10	3.141 587 725 279 96	4.928 309 832 230 581E-06
11	3.141 591 421 504 64	1.232 085 157 898 410E-06
12	3.141 592 345 611 08	3.079 787 163 073 888E-07
13	3.141 592 576 545 07	0.704 478 877 101 905E-08
14	3.141 592 633 463 25	2.012 654 487 515 419E-08

Table 1.

3. Approximating π by excess

Let P_m be a regular polygon of m sides, but circumscribing the unit circle. If A_m is the area of P_m, and Δ_m the area of each triangle over each side of P_m, we see first that

$$A_{2m} = 2m\Delta_{2m} = 2m\tan\alpha_{2m} = 2m\frac{-1+(1+\Delta_m^2)^{1/2}}{\Delta_m} \qquad (2)$$

where

$$\tan\alpha_{2m} = \tan\frac{2\pi}{2m} = \Delta_{2m}$$

as is clear from figure 2. Note that the last equality comes from the relation between the tangent of an angle and the tangent of its half angle:

$$\tan(\alpha/2) = \frac{-1+(1+\tan^2\alpha)^{1/2}}{\tan\alpha}$$

Although the recursion method given by equation (2), approximates the area of the circle, that is π, it does not converge because the numerator

$$-1+(1+\Delta_m^2)^{1/2}$$

tends to zero and the denominator also tends to zero. This method can be easily improved. Multiplying numerator and denominator in equation (2) by:

$$1+(1+\Delta_m^2)^{1/2}$$

we obtain the formula:

$$\Delta_{2m} = \frac{\Delta_m}{1+(1+\Delta_m^2)^{1/2}} \qquad (3)$$

and hence the recursion scheme

$$A_{2m} = 2m\Delta_{2m} = 2m\frac{\Delta_m}{1+(1+\Delta_m^2)^{1/2}} \qquad (4)$$

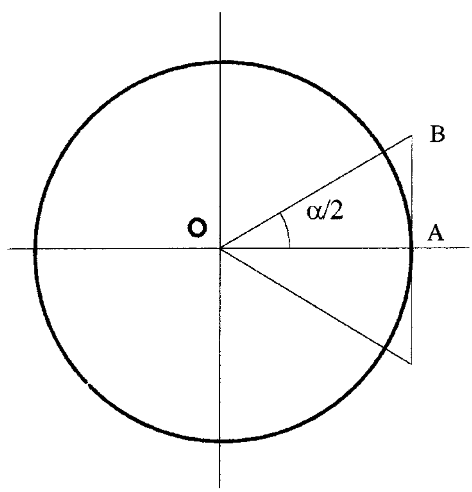

Figure 2.

m	π	Error
2	3.313 708 498 984 76	– 0.172 115 845 394 968
3	3.182 597 878 074 53	– 4.100 522 448 473 631E-02
4	3.151 724 907 429 26	– 1.013 225 383 946 503E-02
5	3.144 118 385 245 87	– 2.525 731 656 073 393E-03
6	3.142 223 629 942 34	– 6.309 763 525 518 263E-04
7	3.141 750 369 169 70	– 1.577 155 799 110 663E-04
8	3.141 632 080 702 25	– 3.942 711 245 574 770E-05
9	3.141 602 510 241 97	– 9.856 652 178 896 751E-06
10	3.141 595 117 718 37	– 2.464 128 572 299 273E-06
11	3.141 593 269 631 75	– 6.160 419 521 172 855E-07
12	3.141 592 808 379 68	– 1.547 898 915 710 277E-07
13	3.141 592 690 962 97	– 3.737 317 610 941 204E-08
14	3.141 592 675 570 45	– 2.198 065 907 066 393E-08
15	3.141 592 690 962 97	– 3.737 317 610 941 204E-08

Table 2.

Table 2 shows the evolution of the error given by approximating π by excess, following the recursion formula (2). Note that the best excess approximation of π, is attained for the 15th iteration, in which the error is about 10^{-8}.

Table 3 shows the evolution of the error given by approximating π by excess,

m	π	Error
2	3.313 708 498 984 76	– 0.172 115 845 394 968
3	3.182 597 878 074 53	– 4.100 522 448 473 542E-02
4	3.151 724 907 429 26	– 1.013 225 383 946 281E-02
5	3.144 118 385 245 90	– 2.525 731 656 111 584E-03
6	3.142 223 629 942 46	– 6.309 763 526 641 809E-04
7	3.141 750 369 168 97	– 1.577 155 791 738 782E-04
8	3.141 632 080 703 18	– 3.942 711 338 966 730E-05
9	3.141 602 510 256 81	– 9.856 667 017 249 520E-06
10	3.141 595 117 749 59	– 2.464 159 797 543 885E-06
11	3.141 593 269 629 31	– 6.160 395 153 997 911E-07
12	3.141 592 807 599 65	– 1.540 098 524 266 397E-07
13	3.141 592 692 092 26	– 3.850 246 210 745 922E-08
14	3.141 592 663 215 41	– 9.625 615 859 931 712E-09
15	3.141 592 655 996 20	– 2.406 404 853 161 348E-09
16	3.141 592 654 191 39	– 6.016 018 794 241 517E-10
17	3.141 592 653 740 19	– 1.504 014 690 567 601E-10
18	3.141 592 653 627 39	– 3.760 103 339 800 480E-11
19	3.141 592 653 599 19	– 9.400 924 483 315 976E-12
20	3.141 592 653 592 14	– 2.351 452 366 156 081E-12
21	3.141 592 653 590 38	– 5.893 063 814 710 331E-13
22	3.141 592 653 589 94	– 1.483 257 960 899 209E-13
23	3.141 592 653 589 83	– 3.819 167 204 710 538E-14
24	3.141 592 653 589 81	– 1.021 405 182 655 144E-14
25	3.141 592 653 589 80	– 3.108 624 468 950 438E-15
26	3.141 592 653 589 79	– 1.776 356 839 400 250E-15

Table 3.

following the recursion formula (4). Note that here the best excess approximation of π, is attained for the 26th iteration, in which the error is about 10^{-15}.

5. Conclusion

The use of approximating polygons for finding some approximations of the number π is explained.

In the second case, the division between two very small numbers has been avoided by means of a simple arithmetic trick.

References

[1] SMITH, J., 1998, *Chicago Tribune Magazine*, August 2.
[2] CRABTREE, D. E., ECCLES, F. M., and SMITH, N. B., 1984, *Precalculus* (Scott, Foresman and Co.).
[3] SPIVAK, M., 1970, *Calculus. Cálculo Infinitesimal* (Reverté S.A.).

A generalization of a result of Brillhart

R. S. MELHAM
School of Mathematical Sciences, University of Technology, Sydney, PO Box 123, Broadway NSW 2007, Australia

(*Received and reviewed by an AEB member*)

Brillhart [1] showed that the sequence $U_n = n^2 - n - 1$ is such that $U_n U_{n+1} = U_{n^2-1}$. The purpose of this paper is to generalize Brillhart's result. Our generalization is contained in the following theorem.

Theorem. Let $U_n = n^2 + an + b$, where a and b are complex numbers. Then

$$U_n U_{n+1} = U_{n^2+(a+1)n+b}$$

Proof. Set $A_n = n^2 + (a+1)n + b$. Then

$$\begin{aligned} U_n U_{n+1} &= [n^2 + an + b][(n+1)^2 + a(n+1) + b] \\ &= [A_n - n][A_n + n + a + 1] \\ &= A_n^2 + aA_n + A_n - (n^2 + (a+1)n) \\ &= A_n^2 + aA_n + b \end{aligned}$$

and this proves the theorem.

With $a = b = -1$ we obtain Brillhart's result.

References

[1] BRILLHART, J., 1964, *Fibonacci Quarterly*, **2**, 220.