©Freund Publishing House Ltd., International Journal of Nonlinear Sciences & Numerical Simulation 10(11-12): 1527-1538, 2009

Binomial Transforms of the k-Fibonacci Sequence

Sergio Falcon and Angel Plaza

Department of Mathematics, University of Las Palmas de Gran Canaria
35017-Las Palmas de Gran Canaria, Spain. sfalcon@dma.ulpgc.es

Abstract

In this paper, we apply the binomial, k-binomial, rising, and falling transforms to the k-Fibonacci sequence. Many formulas relating the so obtained new sequences are presented and proved. Finally, we define and find the inverse transforms of the sequences previously obtained.

Keywords: Binomial, k-Binomial, Rising, and Falling Transforms, k-Fibonacci sequence

1. Introduction

The Fibonacci numbers F_n are the terms of the sequence $\{0,1,1,2,3,5,\ldots\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0=0$, and $F_1=1$. The ratio of two consecutive Fibonacci numbers converges to the Golden Section, $\tau=\frac{1+\sqrt{5}}{2}$, which appears in modern research [1-5], particularly Physics of the high energy particles [6,7] or theoretical Physics [8-11]. Recently, El Naschi has shown that the exceptional Lie symmetry group *E12* together with the compactified Klein modular curve $SL(2,7)_c$ gives $|E12|+|SL(2,7)_c|=685+339=1024$ [12]. Even the so-called Fibonacci-Pascal triangle has been related with the internal symmetries of super strings and 5-Brane in 11 dimensions [13,14].

Fibonacci numbers have been generalized in many ways [2, 15-22]. Gazale [20], Kappraff [21], and later Stakhov[22], Falcon and Plaza [18,19] independently introduced the Generalized Fibonacci numbers of the order k or simply Fibonacci *k*-numbers, depending only on one integer parameter k.

On the other hand, given a sequence, many matrix based transforms can be defined. Some of these transforms are the binomial transform, and other similar formulations, like the rising and falling binomial transforms. The focus of this paper is to apply these transforms to the k-Fibonacci sequence and deduce several relations between the so obtained new sequences.

Given an integer sequence $A=\{a_0,a_1,a_2,\ldots\}$, define the binomial transform B of the sequence A to be the sequence $B(A)=\{b_n\}$, where b_n is given by

$$b_n=\sum_{i=0}^{n}\binom{n}{i}a_i \qquad (1)$$

For any positive integer number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by [18,19]:

$$F_{k,n+1}=k\,F_{k,n}+F_{k,n-1} \quad \text{for } n\geq 1 \qquad (2)$$

with the initial conditions $F_{k,0}=0$, $F_{k,1}=1$.

Note that if "k" is a real variable "x" then $F_{k,n}=F_{x,n}$ and they correspond to the Fibonacci polynomials defined by:

$$F_{n+1}(x)=\begin{cases}1, & n=0\\ x, & n=1\\ xF_n(x)+F_{n-1}(x), & n>1\end{cases}$$

Particular cases of the k-Fibonacci sequences are the classical Fibonacci sequence, for k=1, and the Pell sequence, for k=2.

The first k-Fibonacci sequences indexed in

The On-Line Encyclopedia of Integer Sequences [23], from now on OEIS, are:

$$\{F_{1,n}\}=\{0,1,1,2,3,...\}:A000045$$

$$\{F_{2,n}\}=\{0,1,2,5,12,...\}:A000129$$

$$\{F_{3,n}\}=\{0,1,3,10,33,...\}:A006190$$

$$\{F_{4,n}\}=\{0,1,4,17,72,...\}:A001072$$

$$\{F_{5,n}\}=\{0,1,5,26,135,...\}:A052918$$

$$\{F_{6,n}\}=\{0,1,6,37,228,...\}:A005668$$

1.1 The Pascal 2-triangle.

In [19,24], we created and used the Pascal 2-triangle from the k-Fibonacci sequences as an extension of the classical Pascal triangle, also called the Fibonacci-Pascal triangle [15]. See Table 1.

Table 1

n										**Row sums**
1					1					1
2					1					1
3				1		1				2
4				1		2				3
5			1		3		1			5
6			1		4		3			8
7		1		5		6		1		13
8		1		6		10		4		21
9	1		7		15		10		1	34
10	1		8		21		20		25	55

Among other properties, in this triangle it is verified that, if i indicates the row and j the order of the element $C_{i,j}$ in this row, then $C_{i,j}=C_{i-1,j}+C_{i-2,j-1}$

2. Binomial Transform of the k-Fibonacci Sequences

2.1 Definition 1. *We will indicate the binomial transform of the k-Fibonacci sequence* $\{F_{k,n}\}$ as $B_k=\{b_{k,n}\}$ where:

$$b_{k,n}=\sum_{i=0}^{n}\binom{n}{i}F_{k,i} \tag{3}$$

It is easy to prove that the binomial transform of the classical Fibonacci sequence $\{F_n\}=\{0,1,1,2,3,5,...\}$ is the sequence $B_1=\{b_{1,n}\}=\{0,1,3,8,21,55,...\}$ indexed as A001906 in OEIS and called bisection of classical Fibonacci sequence. Note that this sequence equals sequence of Fibonacci numbers with even indices, $\{b_{1,n}\}=\{F_{2n}\}$.

The only binomial transforms of the k-Fibonacci sequences indexed in OEIS are:

$$B_1=\{0,1,3,8,21,...\}:A001906$$

$$B_2=\{0,1,4,14,48,...\}:A007070\cup\{0\}$$

$$B_3=\{0,1,5,22,95,...\}:A116415\cup\{0\}$$

$$B_4=\{0,1,6,32,168,...\}:A084326$$

Binomial transform of the k-Fibonacci sequences can be obtained as application of the main theorem of this paper that we will prove in the sequel. But previously, we need to demonstrate the following Lemma.

2.1 Lemma. *The binomial transform of the k-Fibonacci sequence verifies the relation*

$$b_{k,n+1}=\sum_{i=0}^{n}\binom{n}{i}\left(F_{k,i+1}+F_{k,i}\right) \tag{4}$$

Proof. Taking into account the addition property of binomial numbers $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$, it is

$$b_{k,n+1}=\sum_{i=0}^{n+1}\binom{n+1}{i}F_{k,i}=\sum_{i=1}^{n+1}\left[\binom{n}{i}+\binom{n}{i-1}\right]F_{k,i}$$
$$=\sum_{i=0}^{n}\binom{n}{i}F_{k,i}+\sum_{i=0}^{n}\binom{n}{i}F_{k,i+1}=\sum_{i=0}^{n}\binom{n}{i}(F_{k,i+1}+F_{k,i})$$

Note that Equation (4) can also be written as $b_{k,n+1}=\sum_{i=0}^{n}\binom{n}{i}F_{k,i+1}+b_{k,n}$.

Now, we proceed to demonstrate the main

theorem of this paper.

2.2 Theorem 1: Main Theorem. *Binomial transform of the k-Fibonacci sequence, $B_k = \{b_{k,n}\}$ verifies the recurrence relation*

$$b_{k,n+1} = (k+2)b_{k,n} - k\,b_{k,n-1} \quad for \;\; n \geq 1 \qquad (5)$$

with initial conditions $b_{k,0} = 0$ and $b_{k,1} = 1$

Proof. With Lemma (1) in mind and formula (1),

$$b_{k,n+1} = \sum_{i=1}^{n}\binom{n}{i}(F_{k,i+1} + F_{k,i}) + (F_{k,1} + F_{k,0}) = \sum_{i=1}^{n}\binom{n}{i}(k\,F_{k,i} + F_{k,i} + F_{k,i-1}) + (F_{k,1} + F_{k,0})$$

$$= (k+1)\sum_{i=1}^{n}\binom{n}{i}F_{k,i} + \sum_{i=1}^{n}\binom{n}{i}F_{k,i-1} + F_{k,1}$$

And therefore,

$$b_{k,n+1} = (k+1)b_{k,n} + \sum_{i=1}^{n}\binom{n}{i}F_{k,i-1} + 1 \qquad (6)$$

On the other hand, taking into account that $\binom{n-1}{n} = 0$, it is

$$b_{k,n} = (k+1)b_{k,n-1} + \sum_{i=1}^{n-1}\binom{n-1}{i}F_{k,i-1} + 1 = k\,b_{k,n-1} + \sum_{i=0}^{n-1}\binom{n-1}{i}F_{k,i} + \sum_{i=1}^{n-1}\binom{n-1}{i}F_{k,i-1} + 1$$

$$= k\,b_{k,n-1} + \sum_{i=1}^{n}\left[\binom{n-1}{i-1} + \binom{n-1}{i}\right]F_{k,i-1} + 1 = k\,b_{k,n-1} + \sum_{i=1}^{n}\binom{n}{i}F_{k,i-1} + 1$$

And, hence,

$$1 + \sum_{i=1}^{n}\binom{n}{i}F_{k,i-1} = b_{k,n} - k\,b_{k,n-1} \qquad (6)$$

Therefore, by substituting this expression in Eq. (6), the looked formula is obtained.

2.3 Theorem 2: Binet's formula for B_k. *The general term of the binomial transform of the k-Fibonacci sequence can be calculated by means of the following Binet-type formula* $b_{k,n} = \frac{r_1^n - r_2^n}{\sqrt{k^2+4}}$ *being* $r_1 = \frac{k+2+\sqrt{k^2+4}}{2}$ *and* $r_2 = \frac{k+2-\sqrt{k^2+4}}{2}$

Proof. The characteristic polynomial equation of recurrence formula (5) is $r^2 - (k+2)r + k = 0$, whose solutions are r_1 and r_2. So, it is $b_{k,n} = C_1 r_1^n + C_2 r_2^n$. Taking into account $b_{k,0} = 0$ and $b_{k,1} = 1$, we deduce this formula.

Note that the characteristic roots verify $r_1 r_2 = k$, and then, the preceding formula can also be written as $b_{k,n} = \frac{r^n - k^n r^{-n}}{r + r^{-1}}$, r being the positive characteristic root, $r = \frac{k+2+\sqrt{k^2+4}}{2} = \sigma + 1$ with $\sigma = \frac{k+\sqrt{k^2+4}}{2}$ as in Eq. (9).

2.4 *Generating function of the binomial transform of the k-Fibonacci sequence.*

Let us suppose the binomial transform of the k-Fibonacci numbers are the coefficients of a power series centered at the origin, and consider the corresponding analytic function $b_k(x)$. This function is called the generating function of the sequence B. So, $b_k(x) = b_{k,0} + b_{k,1}x + b_{k,2}x^2 + \cdots$ And then,

$$(k+2)x\,b_k(x) = (k+2)b_{k,0}x + (k+2)b_{k,1}x^2 + (k+2)b_{k,2}x^3 + \cdots$$

$$k\,x^2 b_k(x) = kb_{k,0}x^2 + kb_{k,1}x^3 + kb_{k,2}x^4 + \cdots$$

From where, since $b_{k,i} = (k+2)b_{k,i-1} - k\,b_{k,n-2}$, $b_{k,0} = 0$, and $b_{k,1} = 1$, we obtain $\left(1-(k+2)x+k\,x^2\right)\; b_k(x) = x$. And, hence, the generating function for the binomial transform of k-Fibonacci sequence $\left\{b_{k,n}\right\}_{n=0}^{\infty}$ is

$$b_k(x) = \frac{x}{1-(k+2)x+k\,x^2}.$$

Function $b_k(x)$ may be also be obtained from the generating function of the *k*-Fibonacci sequence, $f(x) = \frac{x}{1-k\,x-x^2}$ [19]. It is straightforwardly deduced by using the following result proved by Barry and Prodinger, independently [25,26]: If $f(x)$ is the ordinary generating function of the sequence $\left\{a_n\right\}$, then the generating function of the transformed binomial sequence is $\frac{1}{1-x} f\left(\frac{x}{1-x}\right)$.

2.5 Triangle of the binomial transform of the k-Fibonacci sequence.

Following [27,28], we introduce, for each integer *k*, an infinite triangle of numbers T_k, as follows: The left diagonal of the triangle consists of the elements of the *k*-Fibonacci sequence and any number off the left diagonal is the sum of the number to its left and the number diagonally above it to the left.

Then, the sequence on the right diagonal is the binomial transform of the k-Fibonacci sequence.

Every antidiagonal sequence $\left\{a_{k,n}\right\}$ of this triangle verify the same relation as the *k*-Fibonacci initial sequence, that is: $a_{k,n+1} = k\,a_{k,n} + a_{k,n-1}$.

For example, Table 2 shows the triangle T_3 for the 3-Fibonacci sequence and its Binomial Transform:

Table 2

1					0				
2				1		1			
3			3		4		5		
4		10		13		17		22	
5	33		43		56		73		95

Note that the sequence on the right diagonal, $\left\{0,1,5,22,95,409,\cdots\right\}$, is $B_3 = \left\{b_{3,n}\right\}$, precisely the binomial transform of $\left\{F_{3,n}\right\}$.

Sequences from Table 2 indexed in OEIS are:

$\left\{1,4,13,43,142,\cdots\right\}: A003688$,

$\left\{1,4,17,73,314,\cdots\right\}: A018902$,

$\left\{3,13,56,241,1037,\cdots\right\}: A010903, A010920$, and $A095934$

All diagonal sequences verify the relation $a_{3,n+1} = 5a_{3,n} - 3a_{3,n-1}$ while the antidiagonal sequences hold $a_{3,n+1} = 3a_{3,n} + a_{3,n-1}$.

The elements $b_{k,n}$ of the diagonal sequence of order i verify the relation

$$b_{k,n+2} = (k+1)b_{k,n+1} + \sum_{i=0}^{n} b_{k,i} + 1.$$

In a similar way to Spivey and Steil in [26], we now consider three variations of the binomial transform applied to the *k*-Fibonacci sequences.

3. The *k*-binomial transform of the *k*-Fibonacci sequence.

3.1 Definition 2. *The k-binomial transform W of the sequence $\left\{F_{k,n}\right\}$ is the sequence $W_k = \left\{w_{k,n}\right\}$, where $w_{k,n}$ is given by*

$$w_{k,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^n F_{k,i} & \text{for } k \neq 0 \text{ or } n \neq 0 \\ 0 & \text{if } k = 0 \text{ and } n = 0 \end{cases}$$

The 1-binomial transform W_1 coincides with the binomial transform B_1 and is the unique of these sequences indexed in OEIS with the number A001906.

The first k-binomial transforms are:

$W_1 = \{0,1,3,8,21,\cdots\} : A001906$

$W_2 = \{0,2,16,112,768,\cdots\} : 2 \cdot A057084$

$W_3 = \{0,3,45,594,7695,\cdots\}$

$W_4 = \{0,4,96,2048,43008,\cdots\}$

3.2 Triangle of the k-binomial transform.

Create an infinite triangle of numbers so that the left diagonal consists of the k-Fibonacci numbers and any number off the left diagonal is k times the sum of the numbers to its left and diagonally above it to the left. Then the right diagonal is the k-binomial transform W_k of $\{F_{k,n}\}$

This triangle, applied in this case to the 3-Fibonacci sequence is given in Table 3:

Table 3

1	0				
2	1	3			
3	3	12	45		
4	10	39	153	594	
5	33	129	504	1971	7695

The right diagonal sequence, $\{0,3,45,594,7695,99387,\cdots\}$ is precisely the k-binomial transform of the 3-Fibonacci sequence, as in Definition 4.

*3.3 **Theorem 3.** The k-binomial transform of the k-Fibonacci sequence is a recurrence sequence where* $w_{k,n+1} = k(k+2)w_{k,n} - k^3 w_{k,n-1}$ for $n \geq 1$ *with initial conditions* $w_{k,0}=0$ *and* $w_{k,1} = k$

Proof. From the definition of binomial transform and k-binomial transform, we obtain $w_{k,n} = k^n b_{k,n}$. And combining this equation with Eq. (9), relation (8) is obtained.

Similarly to that for the binomial transform, the generating function for the k-binomial transform of the k-Fibonacci sequence is $w_k(x) = \dfrac{kx}{1-k(k+2)x+k^3x^2}$.

This formula can also be obtained by considering that if $f(x)$ is the generating function of sequence F_k, then the generating function of the k-Binomial transform is

$$w_k(x) = \frac{1}{1-kx} f\left(\frac{kx}{1-kx}\right) \text{ [25,27]}.$$

Binet's formula for the k-binomial transform of the k-Fibonacci sequence is $r_{k,n} = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$ being $r_1 = \dfrac{k+2+\sqrt{k^2+4}}{2}k$ and $r_2 = \dfrac{k+2-\sqrt{k^2+4}}{2}k$. It is verified that $r_1 - r_2 = \sqrt{k^2+4}$ and $r_1 = k(\sigma+1)$ where σ is as Eq. (9).

4. The Rising k–Binomial Transform of the k-Fibonacci Sequence

4.1 Definition 3. *The rising k-binomial transform R of the sequence* $\{F_{k,n}\}$ *is the sequence* $R_k = \{r_{k,n}\}$, *where* $r_{k,n}$ *is given by*

$$r_{k,n} = \begin{cases} \sum_{i=1}^{n} \binom{n}{i} k^i F_{k,i} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

The first rising k-Fibonacci sequences indexed in OEIS are

$R_1 = \{0,1,3,8,21,\ldots\} : A001906$

$R_2 = \{0,2,12,70,408,\ldots\} : A001542$

$R_3 = \{0,3,33,360,3927,\ldots\} : A075835$

$R_4 = \{0,4,72,1292,23184,\ldots\} : A060645$

4.2 Triangle of the Rising k-Binomial transform.

Create a triangle of numbers so that the left diagonal consists of the k-Fibonacci numbers and any number off the left diagonal is the sum of the number diagonally above it to the left and k times the number to its left. Then the right diagonal is the rising k-binomial transform R_k of $\{F_{k,n}\}$.

In this case, we can create the triangle in such a way that the left diagonal of order i consists of the k-Fibonacci sequence beginning with the term of order $(2i-1)$.

This triangle, applied to the 3-Fibonacci sequence is shown in Table 4.

Table 4

1					0	[illegible]			
2				1		3			
3			3		10		33		
4		10		33		109		360	
5	33		109		360		1189		3927

The right diagonal sequence of this triangle is $\{0,3,33,360,3927,\ldots\} = F_{3,2n} : A075835$ and coincides with the Rising 3-Binomial transform of $F_{3,n}$. The following diagonal sequence is $\{1,10,109,1189,12970,\ldots\} = F_{3,2n+1} : A078922$

*4.3 **Theorem 4.** The rising k-binomial transform k-binomial transform R_k of the k-Fibonacci sequence $\{F_{k,n}\}$ is the sequence $\{F_{k,2n}\}$ which is called bisection of k-Fibonacci sequence.*

Proof. We have to prove that for any $n \in N$,

$$r_{k,n} = F_{k,2n} \tag{7}$$

Binet's formula for the k-Fibonacci sequences is

$$F_{k,n} = \frac{\sigma^n - (-\sigma)^{-n}}{\sigma + \sigma^{-1}} \tag{8}$$

where σ is the positive characteristic root, $\sigma = \sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$. This number is also called *metallic ratio* or *metallic mean*, and for $k=1$ is precisely the golden ratio [5,29]. Then,

$$r_{k,n} = \sum_{i=1}^{n}\binom{n}{i}k^i F_{k,i} = \sum_{i=1}^{n}\binom{n}{i}k^i \frac{\sigma_1^i - \sigma_2^i}{\sigma_1 - \sigma_2} = \frac{1}{\sigma_1 - \sigma_2}\left(\sum_{i=1}^{n}\binom{n}{i}(k\sigma_1)^i - \sum_{i=1}^{n}\binom{n}{i}(k\sigma_2)^i\right) =$$

$$= \frac{1}{\sigma_1 - \sigma_2}\left((k\sigma_1+1)^n - (k\sigma_2+1)^n\right) = \frac{\sigma_1^{2n} - \sigma_2^{2n}}{\sigma_1 - \sigma_2} = F_{k,2n}$$

since $k\sigma + 1 = \sigma^2$.

Eq. (8) and Eq.(9) give the Binet's formula for the rising k-binomial transform, k-binomial transform R_k of the k-Fibonacci sequence: $r_{k,n} = \frac{r^{2n} - r^{-2n}}{r + r^{-1}}$, where $r = \sigma = \frac{k+\sqrt{k^2+4}}{2}$.

Taking into account the definition of the binomial transform, Eq. (1), the following relation between the terms of the rising k-binomial transform is obtained:

*4.5 **Theorem 5.** The rising k-Fibonacci sequences are recurrence sequences such that*

$$r_{k,n+1} = \left(k^2+2\right)r_{k,n} - r_{k,n-1} \text{ for } n \geq 1, \tag{9}$$

with initial conditions $r_{k,0} = 0$ and $r_{k;1} = k$.

Proof. Taking into account $F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ and formula (7), it is

$$\begin{aligned} F_{k,2n+2} &= k\,F_{k,2n+1} + F_{k,2n} = k\left(k\,F_{k,2n} + F_{k,2n-1}\right) + F_{k,2n} \\ &= \left(k^2+1\right)F_{k,2n} + F_{k,2n} - F_{k,2n-2} = \left(k^2+2\right)F_{k,2n} - F_{k,2n-2} \end{aligned}$$

From where, by Eq. (10), $r_{k,n+1} = \left(k^2+2\right)r_{k,n} - r_{k,n-1}$.

Solving this recurrence equation, the Binet-type formula for the rising k-binomial transform of the k-Fibonacci sequences is obtained:

$$r_{k,n} = \frac{k\left(\sigma_1^n - \sigma_2^n\right)}{\sigma_1 - \sigma_2}$$

with

$$\sigma_1 = \frac{k^2 + 2 + \sqrt{k^4 + 4k^2}}{2} = k\sigma + 1,$$

or $r_{k,n} = \dfrac{\sigma^n - (-\sigma)^{-n}}{\sqrt{k^2 + 4}}$.

Likewise in Theorem 5, it can be proved that the generating function for the rising k-Fibonacci sequence is $r_k(x) = \dfrac{kx}{1 - \left(k^2 + 2\right)x + x^2}$.

5 The Falling k-Binomial Transform of the k-Fibonacci Sequence

5.1 Definition 4. *The falling k-binomial transform F of the sequence* $\{F_{k,n}\}$ *is the sequence* $F_k = \{f_{k,n}\}$, *where* $f_{k,n}$ *is given by*

$$f_{k,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^{n-i} F_{k,i} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

The first falling k-Fibonacci sequences are:

$F_1 = \{0,1,3,8,21,\ldots\} : A001906$

$F_2 = \{0,1,6,29,132,589,\ldots\} : A081179$

$F_3 = \{0,1,9,64,423,2719,\ldots\}$

$F_4 = \{0,1,12,113,984,\ldots\}$

5.2 Theorem 6. *The falling k-Fibonacci sequences verify*
$f_{k,n+1} = 3k f_{k,n} - \left(2k^2 - 1\right) f_{k,n-1}$ for $n \geq 1$
with initial conditions $f_{k,0} = 0$ *and* $f_{k,1} = 1$.

Proof. We first prove that

$$f_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} k^{n+1-i} \left(F_{k,i+1} + F_{k,i}\right) =$$
$$= \sum_{i=0}^{n} \binom{n}{i} k^{n+1-i} F_{k,i+1} + k f_{k,n}.$$

Now the proof is similar to that of Theorem 2.

Analogously to that in Theorem 5, we can prove that the generating function for the falling k-Fibonacci sequence is $f_k(x) = \dfrac{x}{1 - 3k\,x + \left(2k^2 - 1\right)x^2}$. Finally, Binet's formula for the falling k-Fibonacci sequences is $f_{k,n} = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$ where r_1 and r_2 are the characteristic roots. In fact, $r_1 = \dfrac{3k + \sqrt{k^2 + 4}}{2} = \sigma + k$, with σ as in Eq. (11).

5.3 Triangle of the falling k-binomial transform

Create a triangle of numbers so that the left diagonal consists of the k-Fibonacci numbers and any number off the left diagonal is the sum of the number to its left and k times the number diagonally above it to the left. Then the right diagonal is the falling k-binomial transform F_k of $\{F_{k,n}\}$.

Table 5 shows the triangle for the 3-Fibonacci sequence.

Table 5

1					0					
2				1		1				
3			3		6		9			
4		10		19		37		64		
5	33		63		120		231		423	

The right diagonal sequence, $\{0,1,9,64,423,2719,\ldots\}$ is precisely the sequence F_3.

Evidently, if $k = 1$, the falling 1-binomial transform coincides with the 1-binomial transform. And taking into account all the notes indicated in the preceding sections,

$$B_1 = W_1 = R_1 = F_1 : A001906 \qquad (10)$$

Let T^m denote m successive applications of the transform T; i.e., $T^m(F_k) = T(T(\cdots T(F_k)\cdots))$ where T occurs m times.

From Eq. (11) we obtained

$B_1^2 = W_1^2 = R_1^2 = F_1^2 : A093131$,

$B_1^3 = W_1^3 = R_1^3 = F_1^3 : A099453 \cup \{0\}$,

$B_1^4 = W_1^4 = R_1^4 = F_1^4 : A081574$, etc.

In [27, Theorem 2.2] , it is proved that $B^k = F_k$. There are many relations between transforms B, W, R, and F. In general, for $k \neq 1$, the composition of two of these transforms is not commutative. On the other hand, the composition of two of them may be equal to the composition of two other transforms. Next, some of these relations are proved.

5.4 Theorem 7 *The following relations hold:*

a) $RB = FR$
b) $RB = BW$
c) $FB = BF$

Proof. (a) It is enough to prove that $(rb)_n = (fr)_n$. From the definitions of B and R transforms, the general term of the sequence RB is

$$(rb)_n = \sum_{i=0}^{n}\binom{n}{i}k^i\sum_{j=0}^{n}\binom{i}{j}F_{k,j} = \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}k^iF_{k,j} = \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{j}\binom{n-j}{i-j}k^iF_{k,j}$$
$$= \sum_{j=0}^{n}\binom{n}{j}F_{k,j}\sum_{i=j}^{n}\binom{n-j}{i-j}k^i = \sum_{j=0}^{n}\binom{n}{j}F_{k,j}\sum_{l=0}^{n-j}\binom{n-j}{l}k^{l+j} = \sum_{j=0}^{n}\binom{n}{j}k^j(k+1)^{n-j}F_{k,j}$$

The third equality holds by trinomial revision [30, p.174] and the last one by the binomial theorem.

On the other hand, the general term of the sequence FR is

$$(fr)_n = \sum_{i=0}^{n}\binom{n}{i}k^{n-i}\sum_{j=0}^{i}\binom{i}{j}k^jF_{k,j} = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}k^{n+j-i}F_{k,j} = \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{j}\binom{n-j}{i-j}k^{n+j-i}F_{k,j}$$
$$= \sum_{j=0}^{n}\binom{n}{j}F_{k,j}k^{n+j}\sum_{i=j}^{n}\binom{n-j}{i-j}k^{-i} = \sum_{j=0}^{n}\binom{n}{j}F_{k,j}k^n\sum_{l=0}^{n-j}\binom{n-j}{l}\left(\frac{1}{k}\right)^{l+j}$$
$$= \sum_{j=0}^{n}\binom{n}{j}k^n\left(\frac{1}{k}+1\right)^{n-j}F_{k,j} = \sum_{j=0}^{n}\binom{n}{j}k^j(k+1)^{n-j}F_{k,j}$$

Proof. (b) The general term of the sequence BW is

$$(bw)_n = \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}k^iF_{k,j} = \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}k^iF_{k,j} = (rb)_n$$

Proof. (c) Following a similar argument to that for part *(a)*, we have:

$$(fb)_n=\sum_{i=0}^{n}\binom{n}{i}k^{n-i}\sum_{j=0}^{i}\binom{i}{j}F_{k,j}=\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}k^{n-i}F_{k,j}=\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{j}\binom{n-j}{i-j}k^{n-i}F_{k,j}$$
$$=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}\sum_{i=j}^{n}\binom{n-j}{i-j}k^{n-i}=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}\sum_{l=0}^{n-j}\binom{n-j}{l}k^{n-l-j}=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}k^{n-j}\left(\frac{1}{k}+1\right)^{n-j}$$
$$=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}\,(k+1)^{n-j}$$

On the other hand,

$$(bf)_n=\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}k^{i-j}F_{k,j}=\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}k^{i-j}F_{k,j}=\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{j}\binom{n-j}{i-j}k^{i-j}F_{k,j}$$
$$=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}\sum_{l=0}^{n-j}\binom{n-j}{l}k^{l}=\sum_{j=0}^{n}\binom{n}{j}F_{k,j}\,(k+1)^{n-j}$$

and the theorem is proved.

5.5 Theorem 8. *For any* $m\in N$, *it is* $FB^m=B^mF$.

Proof. By induction: For $m=1$ it is the preceding property. Let us suppose the property holds for $m-1$: $FB^{m-1}=B^{m-1}F$. Then:

$$FB^m=\left(FB^{m-1}\right)B=\left(B^{m-1}F\right)B=B^{m-1}\left(FB\right)$$
$$=B^{m-1}\left(BF\right)=B^mF$$

All the binomial transforms of the classical Fibonacci sequence, that is $k=1$, are equal.

In Table 6 these transformations are denoted by T_k^m, where T is B, W, R and F, for $k=1,2,3,4$ and $n=1,2,3$.

Table 6

Tr.	Power **1**	**2**	**3**
T_1	$A001906$	$A093131$	$A099453$
B_2	$A007070\cup\{0\}$	$A080179$	$\{0,1,8,50,276,\ldots\}$
W_2	$\{0,2,16,112,768,\ldots\}$	$\{0,4,64,896,12288,\ldots\}$	$\{0,8,256,7168,\ldots\}$
R_2	$A001542$	$\{0,4,56,716,9072,\ldots\}$	$\{0,8,240,6424,\ldots\}$
F_2	$A081179$	$A091182$	$A081184$
B_3	$A116415\cup\{0\}$	$A083327$	$\{0,1,9,64,423,\ldots\}$
W_3	$\{0,3,45,594,7695,\ldots\}$	$\{0,27,3645,433026,\ldots\}$	$\{0,8,256,7168,\ldots\}$
R_3	$\{0,3,33,360,3927,\ldots\}$	$A075835$	$\{0,27,2889,295812,\ldots\}$
F_3	$\{0,1,9,64,423,2719,\ldots\}$	$\{0,1,15,172,1785,\ldots\}$	$\{0,1,141,5058,\ldots\}$
B_4	$A084326\cup\{0\}$	$A091870$	$A093145$
W_4	$\{0,4,96,2048,43008,\ldots\}$	$\{0,16,1536,131072,\ldots\}$	$\{0,64,24576,838860,\ldots\}$
R_4	$A060645$		
F_4	$\{0,1,12,113,984,\ldots\}$		

6 The Inverse Binomial Transform

There are also some nice relationships between the binomial transform and the inverse binomial transform which is defined as follows.

6.1 Definition 5. *Given a sequence $A=\{a_0,a_1,a_2,\ldots\}$, the inverse binomial transform of A is defined [26,27,31,32] to be a sequence $B^{-1}(A)=\{c_n\}$, where c_n is given by*

$$c_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a_i \tag{11}$$

It is easy to see that $B^{-1}(B(A))=B(B^{-1}(A))=A$. Spivey and Steil show [27, Theorem 3.2] that, if $k\in Z$ then $B^k(A)=F(A,k)$. In other words, k successive applications of the binomial transform (or the inverse binomial transform, if $k\in Z^-$) is equivalent to the falling k-binomial transform. In the same paper, it is proved that $W(A,k)=F(R(A,k),k-1)$, that is, the k-binomial transform is equivalent to the composition of the falling $(k-1)$-binomial transform with the rising k-binomial transform.

In particular, the inverse of the preceding transforms are:

$$F_{k,n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}b_{k,i},$$

$$F_{k,n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}k^{-i}w_{k,i},$$

$$F_{k,n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}k^{-n}r_{k,i};$$

$$F_{k,n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}k^{n-i}f_{k,i}$$

6.2 Matrix form of the binomial transforms.

Let us consider the sequences of k-Fibonacci numbers in matrix form as $S=(F_{k,0},F_{k,1},F_{k,2},\ldots)^T$, the diagonal matrix $K=diag(k^0,k^1,k^2,\ldots)^T$, and the Pascal matrix $P=\begin{pmatrix}1&0&0&0&\ldots\\1&1&0&0&\ldots\\1&2&1&0&\ldots\\1&3&3&1&\ldots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$.

Finally, let us consider the infinite matrices of binomial transforms of k-Fibonacci sequences,

$B=(b_{k,0},b_{k,1},b_{k,2},\ldots)^T$,

$W=(w_{k,0},w_{k,1},w_{k,2},\ldots)^T$,

$R=(r_{k,0},r_{k,1},r_{k,2},\ldots)^T$,

and

$F=(f_{k,0},f_{k,1},f_{k,2},\ldots)^T$,

respectively for the binomial, k-binomial, rising and falling transforms.

The following relations hold:

Binomial Transform:	$B=P\cdot S$
k-Binomial Transform:	$W=K\cdot P\cdot S$
Rising Binomial Transform:	$R=P\cdot K\cdot S$
Falling Binomial Transform:	$F=K\cdot P\cdot K^{-1}\cdot S$

Both matrices P and K are invertible: the inverse of P is matrix P^{-1} of entries $(-1)^{i-j}\binom{i}{j}$ and the inverse of K is the diagonal matrix $K^{-1}=diag(k^0,k^{-1},k^{-2},\ldots)$. Then, we can obtain the expressions for the matrix of the k-Fibonacci numbers S in several ways:

$$S=P^{-1}\cdot B=P^{-1}\cdot K^{-1}\cdot W=K^{-1}\cdot P^{-1}\cdot R=K\cdot P^{-1}\cdot K^{-1}\cdot F,$$

which coincide, evidently, with the respective equations of the preceding sections.

Conclusions.

In this paper the binomial, k-binomial, rising, and falling transforms have been applied to the k-Fibonacci sequences. Many relations between the new sequences so obtained have been proved. Finally, we have defined and found the inverse transforms of the preceding sequences.

Acknowledgements

This work has been supported in part by CICYT Project number MTM2008-05866-C03-02 from Ministerio de Educación y Ciencia of Spain.

Bibliography.

[1] Hoggat VE. "Fibonacci and Lucas numbers". Palo Alto, CA: Houghton-Mifflin; 1969.

[2] Horadam AF. "A generalized Fibonacci sequence". Math Mag 1961;68: 455-9.

[3] Vajda S. "Fibonacci & Lucas Numbers, and the Golden Section. Theory and Applications". Ellis Horwood limited; 1989.

[4] Stakhov A." The Generalized Principle of the Golden Section and its applications in mathematics, science, and engineering". Chaos, Solitons & Fractals 2005;26(2): 263-289.

[5] Spinadel VW. "The metallic means family and forbidden symmetries". Int Math J 2002;2(3): 279-88.

[6] El Naschie MS. "Modular groups in Cantorian $E^{(\infty)}$ high-energy physics". Chaos, Solitons & Fractals 2003;16(2): 353-66.

[7] El Naschie MS. "Experimental and theoretical arguments for the number and mass of the Higgs particles". Chaos, Solitons & Fractals 2005;23(4): 1091-8.

[8] El Naschie MS. "The Golden Mean in Quantum Geometry, KnotTheory and Related Topics". Chaos, Solitons, & Fractals 1999; 10(8): 1303-1307.

[9] El Naschie MS. "Notes on superstrings and the infinite sums of Fibonacci and Lucas numbers". Chaos, Solitons & Fractals 2001;12(10): 1937-40.

[10] El Naschie MS. "Elementary prerequisites for E-infinity theory". Chaos, Solitons & Fractals 2006;30(3): 579-605.

[11] El Naschie MS. "Fuzzy knot theory interpretation of Yang-Mills instantons and Witten's 5-Brane model". Chaos, Solitons & Fractals 2008;38(5): 1349-54.

[12] El Naschie MS. "On the Witten—Duff five Branes model together with knots theory and E8E8 super strings in a single fractal spacetime theory". Chaos, Solitons \& Fractals (2008), doi:10.1016/j.chaos. 2008.08.005

[13] Elokaby A. "On the Fibonacci origin of the internal symmetries of super strings and 5-Brane in 11 dimensions". Chaos, Solitons & Fractals (2008), doi:10.1016/j.chaos. 2008.09.025.

[14] Elokaby A. "Knot wormholes and the dimensional invariant of exceptional Lie groups and Stein space hierarchies". Chaos, Solitons & Fractals (2008), doi:10.1016/j.chaos.2008.07.003.

[15] Kilic E, Stakhov AP. "On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices". Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos. 2007.10.007.

[16] Kocer EG, Tuglu N, Stakhov A. "On the m-extension of the Fibonacci and Lucas p-numbers". Chaos, Solitons & Fractals (2007), doi:10.1016/j.chaos.2007.09.071.

[17] Kilic E. "The Binet formula, sums and representations of generalized Fibonacci p-numbers". European Journal of Combinatorics 2008;29(3): 701–11.

[18] Falcon S, Plaza A. "On the Fibonacci k-numbers". Chaos, Solitons & Fractals 2007; 32(5): 1615-24.

[19] Falcon S, Plaza A. "The k-Fibonacci sequence and the Pascal 2-triangle". Chaos, Solitons & Fractals 2007;33(1): 38-49.

[20] Gazale MJG. "From pharaons to fractals". Princeton, NJ: Princeton University Press; 1999.

[21] Kappraff J. "Beyond measure. A guided tour through nature, myth, and number". Singapore: World Scientific; 2002.

[22] Stakhov AP. "Gazale formulas, a new class of the hyperbolic Fibonacci and Lucas functions, and the improved method of the ''golden'' cryptography". Moscow: Academy of Trinitarism, No. 77-6567, publication 14098, 21.12.2006. http:// www.trinitas.ru/rus/doc/0232/004a/023210 63.htm.

[23] Sloane NJA."The On-Line Encyclopedia of

Integer Sequences" (2006), www.research.att.com/~njas/sequences/.

[24] Falcon S, Plaza A. "The k-Fibonacci polynomials and their derivatives". Chaos, Solitons & Fractals(2007), doi:10.1016/j.chaos.2007.03.007.

[25] Barry P. "On Integer-Sequence-Based Constructions of Generalized Pascal Triangles". Journal of Integer Sequences 2006;9(art. 06.2.4): 1-34.

[26] Prodinger H. "Some information about the binomial transform". The Fibonacci Quarterly 1994;32(5): 412-415.

[27] Spivey MZ, Steil LL. "The k-Binomial Transforms and the Hankel Transform". Journal of Integer Sequences 2006;9(art. 06.1.1): 1-19.

[28] Chen KW, "Identities from the binomial transform". Journal of Number Theory 124 (2007) 142-150.

[29] Falcon S, Plaza A. "The metallic ratios as limits of complex valued trans-formations". Chaos, Solitons & Fractals (2008), doi:10.1016/j.chaos.2007.11.011

[30] Graham RL, Knuth DE, Patashnik O. "Concrete Mathematics". Addison Wesley Publishing Co.; 1998.

[31] Flajolet P. "On congruences and continued fractions for some classical combinatorial quantities". Discrete Math. 1982;41: 145-153.

[32] Knuth DE. "The Art of Computer Programming Vol. 3". Addison-Wesley, Reading, MA, 1973.