



Longest-edge n -section algorithms: Properties and open problems



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ABSTRACT

In this paper we survey all known (including own recent results) properties of the longest-edge n -section algorithms. These algorithms (in classical and recently designed conforming form) are nowadays used in many applications, including finite element simulations, computer graphics, etc. as a reliable tool for controllable mesh generation. In addition, we present a list of open problems arising in and around this topic.

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1. Introduction

In this paper, the longest-edge (LE) n -section is generally understood as splitting the simplex (segment, triangle, tetrahedron, etc.) towards its longest edge (or any of them if there are several such edges) in n subsimplices, and, thus, requires a positioning $n - 1$ points on the longest edge, see Fig. 1 as an illustration for $n = 2$ and Fig. 5—for $n = 3$ and 4.

The simplest case when $n = 2$, and the midpoint of the longest edge is used, leads to the family of so-called *bisection algorithms*, which were originally used for solving nonlinear equations, see e.g. [1–3] and references therein. As the finding (or, actually, estimation of) the roots of such equations required some knowledge on the partitions used, some important geometric properties of partitions generated by such algorithms were analysed and proved in a number of works in 70th, see [4–8]. Later, since the mid of eighties, mainly due to efforts of M. C. Rivara, bisection-type algorithms became popular also in finite element method (FEM) community for mesh refinement/adaptation purposes [9–13]. The bisection methods are nowadays widely used in computer graphics [14,15], e.g. in rendering and level of details [16–18], in terrain modelling—for constructing digital elevation models [19–21]. Also, in scientific visualization problems, data structures [16,22], and even for space-filling curve and domain decomposition [23].

In two-dimensional setting, the classical longest-edge (LE-) bisection algorithm bisects simultaneously all triangles by medians to the longest edge of each triangle in a given triangulation. In this way, an infinite sequence of nested triangulations

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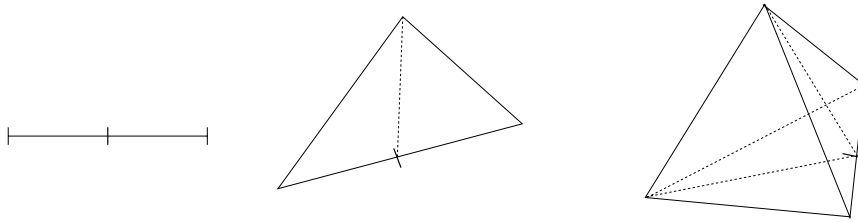


Fig. 1. LE n -sections with $n = 2$ for space dimensions one, two, and three.



Fig. 2. The classical LE-bisection algorithm produces, in general, nonconforming partitions.

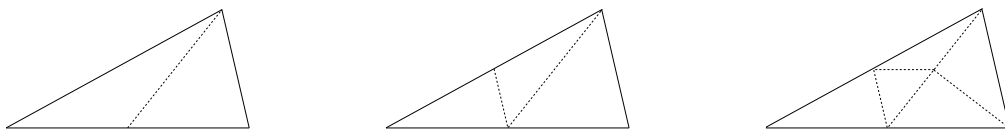


Fig. 3. A conforming LE-bisection algorithm always produces conforming partitions.

can be generated. However, this type of refinements may lead, in general, to the so-called hanging nodes and thus refined triangulations may not all be conforming, in general (see Fig. 2). However, many real-life applications where triangulations are used, e.g. the calculations by FEM, require the property of conformity [24]. Therefore, a modified version of the classical LE-bisection was recently introduced in [25], where only elements sharing the longest edge of the whole triangulation are bisected at each step (see Fig. 3). In this way, all produced triangulations are conforming a priori and such a version was called the conforming LE-bisection algorithm. Obviously, both above variants of the algorithm are applicable (in the same manner) to simplicial partitions in any dimension, see the next sections for the exact definitions and results obtained so far in this case. It is worth to mention that a practical realization of bisection algorithms is, in principle, much simpler than red, red–green, and green refinements of simplices to provide mesh conformity, especially in the case of local mesh refinements and in three or higher dimensions (see [26,27]).

The longest-edge n -section algorithms with larger values of n —see Fig. 5, as potentially suitable techniques for generation of anisotropic meshes used in many real-life applications have attracted much attention in the last five years. First mathematical results on properties of triangular partitions generated by them have been reported in [28,29] (see also [30]). These results and many very recently obtained ones in this direction will be presented and discussed in what follows as well.

2. Basic terminology and definitions

Let the symbol d stand for the space dimension. The convex hull of $d + 1$ points in \mathbf{R}^d for $d \in \{1, 2, 3, \dots\}$, which are not contained in a hyperplane of dimension $d - 1$, is called a d -simplex or just a *simplex* (also, we will be using classical terms – triangle for $d = 2$ and tetrahedron – in the case $d = 3$). The symbol S (possibly with subindices) will be generally used for simplices. The angles between its $(d - 1)$ -dimensional faces (referred to as *facets*) are called *dihedral*. In what follows, $\overline{\Omega} \subset \mathbf{R}^d$ always denotes a bounded polytope.

Definition 1. A partition of $\overline{\Omega}$ into a finite number of simplices such that their union is $\overline{\Omega}$ and any two simplices have disjoint interiors is called a simplicial partition \mathcal{T}_h over $\overline{\Omega}$. If, in addition, any facet of any simplex from \mathcal{T}_h is a facet of another simplex from \mathcal{T}_h or belongs to the boundary $\partial\Omega$ of $\overline{\Omega}$, \mathcal{T}_h is called a conforming (or face-to-face) simplicial partition.

Definition 2. The discretization parameter h is the length of the longest edge in a given simplicial partition.

Remark 1. Sometimes, for partitions which are not necessarily conforming, the term “dissection” is used, e.g. in discrete and computational geometry. And, in the case, of a conforming partition one often says that we have a mesh, which is the term commonly used in FEM community.

Definition 3. The set of simplicial partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h \rightarrow 0}$ is said to be a family of simplicial partitions if for any $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

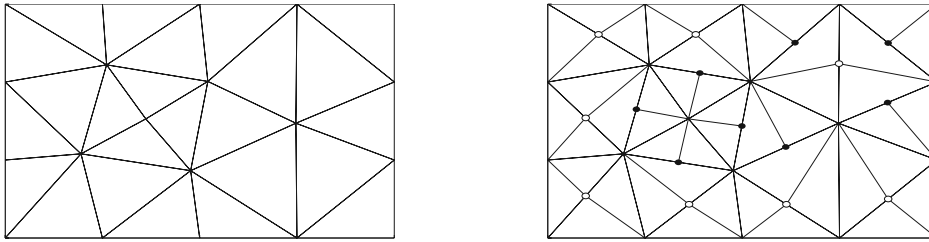


Fig. 4. Performance of LE n -section algorithm for $n = 2$ (bisection).

In [31,32] the following *minimum angle condition* was introduced for triangulations: there should exist a constant α_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$0 < \alpha_0 \leq \alpha_K, \tag{1}$$

where α_K is the minimal angle of K .

Later condition (1) was weakened in [33–35] (see also [36]) and the so-called *maximum angle condition* was proposed: There exists a constant γ_0 such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any triangle $K \in \mathcal{T}_h$ we have

$$\gamma_K \leq \gamma_0 < \pi, \tag{2}$$

where γ_K is the maximum angle of K .

Remark 2. Condition (1) obviously implies (2), since $\gamma_K \leq \pi - 2\alpha_K \leq \pi - 2\alpha_0 =: \gamma_0$, but the converse implication does not hold.

Definition 4. A family $\mathcal{F} = \{\mathcal{T}_h\}_{h \rightarrow 0}$ of simplicial partitions of a bounded polytope $\overline{\Omega}$ is called *regular* if there exists a constant $C > 0$ such that for all partitions $\mathcal{T}_h \in \mathcal{F}$ and for all simplices $S \in \mathcal{T}_h$ we have

$$\text{meas}_d S \geq C(\text{diam } S)^d, \tag{3}$$

where meas_d stands for the d -dimensional measure.

It can be easily checked that for $d = 2$ condition (3) is equivalent to condition (1). Some other definitions of regularity for simplicial partitions equivalent to Definition 4 can be found in [37–39].

Definition 5. A family $\mathcal{F} = \{\mathcal{T}_h\}_{h \rightarrow 0}$ of simplicial partitions of a bounded polytope $\overline{\Omega}$ is called *strongly regular* if there exists a constant $C > 0$ such that for all partitions $\mathcal{T}_h \in \mathcal{F}$ and for all simplices $S \in \mathcal{T}_h$ we have

$$\text{meas}_d S \geq Ch^d. \tag{4}$$

Remark 3. Under the above regularity conditions various a priori error estimates and convergence results for FEMs applied to elliptic (and also parabolic) problems are usually obtained [24].

Now we give the exact definitions and illustrations for two most popular variants of the LE n -section algorithm. It is, in fact, sufficient to describe only one step for each of the variants.

One step of the *longest-edge (LE) n -section algorithm* is defined as follows (see Fig. 4 for a sample illustration):

- (I) choose the longest edge in each simplex of a given simplicial partition and split it by $n - 1$ points into n subedges of equal length;
- (II) split each simplex in the partition into n subsimplices using $n - 1$ points lying on its longest edge.

Remark 4. Note that hanging nodes appearing in this algorithm may be avoided by extending the subsidiary-induced refinement from the target subdivided elements. Then, a single cell partition can propagate LE subdivisions over paths emanating from the target initial cell, see e.g. [40–42] for details. It should be noted that a variant of these methods also produces mesh conformity. See, for example [9], where mesh conformity is tackled by such a propagation scheme.

It should be noted that when introducing nodes in the LE n -section as a matter of the refinement process, this yields hanging nodes. Rather than simply joining new edge nodes to the opposite vertices which may yield small angles in some case, the variant of Remark 4 enforces a consistent LE strategy that may consequently propagate some distance into the mesh. The propagation concludes with a terminating triangle pair that share their respective longest edges or the last triangle has its longest-edge at the boundary, see [22,42].

One step of the *conforming (face-to-face) longest-edge (LE) n -section algorithm* is defined as follows:

- (I) choose the longest edge ℓ in a given face-to-face simplicial partition and split it by $n - 1$ points into n subedges of equal length;
- (II) split each simplex S sharing ℓ into n subsimplices by hyperplanes passing through one of the above mentioned $n - 1$ points (lying on ℓ) and vertices of S that do not belong to ℓ .

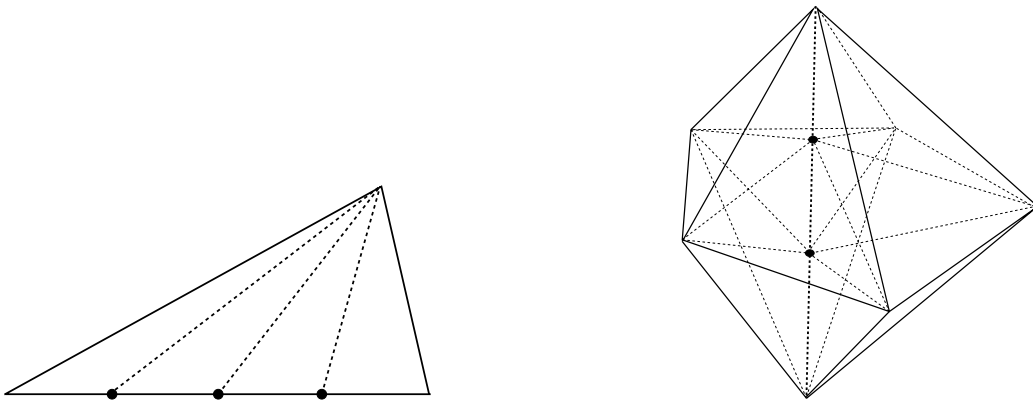


Fig. 5. LE 4-section of a triangle with three inserted points on its longest edge (left). The trisection ($n = 3$) of tetrahedra around the longest edge ℓ with two points on it marked (right).

Remark 5. From the step (II) in above, we observe that all simplicial partitions generated by the conforming LE n -section algorithm remain face-to-face and the refinement is local in any space dimension. See Fig. 5 for an illustration of the case $n = 3$ (trisection) for tetrahedral partitions.

Remark 6. We emphasize that while the classical LE n -section algorithm can be actually applied to any (initial) simplicial partition, not necessarily conforming one, the conforming LE n -section algorithm can only be applied to conforming (initial) simplicial partitions.

Remark 7. It is always possible to construct an initial simplicial partition of any bounded polytopic domain, even conforming one. For example, the idea of [43] with such a construction for an arbitrary bounded polyhedron, can be naturally extended to the case of bounded polytopes.

3. Main results

In this section we will summarize in a compact form all available mathematical and numerical results obtained for the classical and conforming LE n -section algorithms for various values of n and for different dimensions.

3.1. Classical LE n -section algorithm

In many real-life applications it is necessary not only be able to generate some simplicial partitions of a given polytopic domain but also be able to refine this domain in simplices of infinitely small sizes in a certain (well-defined and easy-to-code) way. This is needed e.g. in FEM domain and simulations, for finding roots of equations, in computer graphics, etc. In mathematical terms it is formulated so that the corresponding discretization parameter of simplicial partitions produced tends to zero, and, thus, a family of partitions is generated. The LE bisection algorithm is one of the most suitable techniques for this purpose as the following theorem holds.

Theorem 1. Let $n = 2$ and $d \in \{2, 3, \dots\}$. Then the classical LE n -section algorithm produces the family of simplicial partitions for any polytopic domain $\Omega \subset \mathbf{R}^d$.

In the proof of the theorem we use the idea of constructing the initial partitions mentioned in Remark 7, and then the result of Kearfott [5]. Moreover, even a computable estimate for the rate of decay of the corresponding discretization parameter is derived in [5], which is very important for reliable controlling the mesh characteristics during the refinement process. The estimate of Kearfott was later considerably improved for the two-dimensional case by Stynes in [7,8] (see also the work by Adler [4]). Very recently, similar estimates for the size of elements produced were derived for $n = 3$ (trisections) in [44], and for $n \geq 4$ in [45] by the authors of this work, however, only in the two-dimensional case.

Besides providing the arbitrary smallness of the partitions generated one is often interested in various regularity properties, for example in validity of angle condition (1). In this respect, the following key results are available by now.

Theorem 2. Let $n \in \{2, 3\}$ and $d = 2$. Then the classical LE n -section algorithm produces the family of triangulations where any angle α of any triangle from any triangulation in the family is such that

$$\alpha \geq \frac{\alpha_0}{c}, \quad (5)$$

where α_0 is the smallest angle in the initial triangulation, and c is some positive constant.

For the proofs in case of $n = 2$ see [6] (and a more recent paper [46]), the case $n = 3$ is considered in [47,28]. In both cases, the concrete (and optimal) values of the constants c are derived: $c = 2$ for $n = 2$ and $c = \frac{\pi/3}{\arctan(\sqrt{3}/11)}$ for $n = 3$. The case $n = 3$ is analysed using the concept of the Poincaré half-plane with the hyperbolic distance d between points z_1 and z_2 defined by the formula $\cosh d = 1 + \frac{|z_1 - z_2|^2}{2 \cdot \operatorname{Im} z_1 \cdot \operatorname{Im} z_2}$; and the following property of the functions W defining the LE 3-section partition: for any z_1 and z_2 in the space of triangles one gets $d(W(z_1), W(z_2)) \leq d(z_1, z_2)$, see [47] for more details.

Even a stronger result on regularity properties can be obtained, see the next theorem.

Theorem 3. *Let $n = 2$ and $d = 2$. Then the classical LE bisection algorithm produces only a finite number of different triangular shapes.*

For the classical proofs, based on the Euclidean geometry, see the works [7,8] and [4]. A new proof of this result, based on hyperbolic metric, was recently presented by the authors in [46], by demonstrating that the orbit of any initial triangle is always finite.

It is quite obvious that increasing the value n , one could lose the regularity of partitions produced as the resulting simplices become thinner. In this respect, we have the following important result, the main consequence of which is to use the LE n -sections with large n 's carefully.

Theorem 4. *Let $n \geq 4$ and $d = 2$. The iterative application of the classical LE n -section algorithm to any given triangle always generates a sequence of subtriangles whose minimum angles tend to zero.*

The proof of the above theorem, presented in [29], is based on finding an infinite sequence of triangles among all generated triangles whose minimum angles tend to zero. For this purpose, in each iteration of the LE n -section we pick up the subtriangle with the minimum angle. In [48] the same result is proved by using different arguments: a normalization process and elementary complex variable functions.

Remark 8. On the base of the result of Theorem 4, it was recently shown in [49], that the LE n -section algorithm ($n \geq 4$) applied to any d -simplex in \mathbf{R}^d ($d > 2$), produces a sequence of simplicial meshes with minimum interior solid angles converging to zero. The result is based on noting that any simplicial solid angle having a null angle between two edges has zero measure.

Remark 9. One remarkable property of the classical LE bisection (i.e. $n = 2$) algorithm in the two-dimensional case is the self-improvement of the quality of triangulations: the percentage of quasi-equilateral triangles (and the area covered by these triangles) increases as the bisection refinement proceeds. Thus, this refinement scheme improves angles [50], and such an improvement effect has been studied in depth in [51,11].

3.2. Conforming LE n -section algorithm

The conforming version of the LE n -section technique was introduced in [25] and further developed in [52,53] for $n = 2$, and naturally generalized (and analysed for some properties) to any n in the recent work by the authors [30]. Many results similar to those of the classical version can be obtained independently, e.g. as in [52], however, the following observation can be of a great help and allow to get certain results for this new version relatively easily.

Theorem 5. *Applying the conforming LE n -section algorithm to some initial conforming simplicial partition, one does not produce any simplicial shape different from all the simplicial shapes produced by the classical variant of the LE n -section algorithm applied to the same initial conforming simplicial partition.*

The proof is rather straightforward as in both variants we are splitting each simplex just in the same way—towards its longest edge.

Theorem 6. *Let $n \in \{2, 3, \dots\}$ and $d \in \{2, 3, \dots\}$. Then the conforming LE n -section algorithm produces the family of simplicial partitions for any polytopic domain $\Omega \subset \mathbf{R}^d$.*

The above theorem can be proved in the same way as similar theorems in [52,30] proved there for the two-dimensional case.

Theorem 7. *Let $n \in \{2, 3\}$, and $d = 2$. Then the conforming LE n -section algorithm produces the family of triangulations where any angle α of any triangle from any triangulation in the family is such that*

$$\alpha \geq \frac{\alpha_0}{c}, \quad (6)$$

where α_0 is the smallest angle in the initial triangulation, and c is some positive constant.

The proof immediately follows from [Theorems 2](#) and [5](#). For $n = 2$ and $d = 2$, the above result was also proved in [\[25,52\]](#) using another arguments.

Theorem 8. *Let $n = 2$ and $d = 2$. Then the conforming LE bisection algorithm produces only a finite number of different triangular shapes.*

The proof immediately follows from [Theorems 3](#) and [5](#).

Theorem 9. *Let $n \geq 4$ and $d = 2$. The iterative application of the conforming LE n -section algorithm to any given triangle always generates a sequence of subtriangles with their minimum angles tending to zero.*

For the proof see [\[30\]](#). [Theorem 9](#) can also be proved by arguments from [\[49\]](#).

Remark 10. In fact, from the proofs of [Theorems 4](#) and [9](#) it follows that for both variants – classical and conforming ones – one can always find sequences of triangles with their maximum angles approaching π , thus breaking for both cases even the weaker regularity requirement—the maximum angle condition [\(2\)](#).

Another interesting property of the conforming version is proved in [\[54\]](#).

Theorem 10. *Let $n = 2$ and $d \in \{2, 3, \dots\}$. If $\mathcal{F} = \{\mathcal{T}_h\}_{h \rightarrow 0}$ is a family of simplicial partitions of $\overline{\Omega}$ generated by the conforming LE n -section algorithm, then \mathcal{F} is regular if and only if it is strongly regular.*

Remark 11. In most of practical problems (due to limits of resources) various adaptive procedures are used, where we refine only a part of simplices in the mesh, and do not refine the rest of them. In this situation, a very hard problem of so-called “hanging objects” (nodes, lines, hyperfaces, etc.) in higher dimensions naturally appears. One promising approach to avoid this problem at all (and in any dimension) is to use the blend of a suitable mesh density function, defining the desired sizes of elements over the solution domain (it can be constructed e.g. on the base of a posteriori error estimates), and some conforming LE n -section algorithms, see [\[52,53\]](#) for details and also for many numerical experiments with this idea for problems with potential singularities at various reentrant corners, along boundary and interior layers, and around small inclusions.

3.3. Numerical experiments with LE bisections in 3D

Most of the theorems presented in this section are limited to the two-dimensional case, and not so many mathematical results for LE n -sections are obtained in dimensions three and higher, see e.g. works [\[54–56,12\]](#) devoted to “3D bisection-case”.

Many reasons complicating the analysis of tetrahedral partitions and their LE refinements are given in the above mentioned works, among those we could still emphasize that the 3D geometry “behaves” quite differently from the planar geometry (e.g. there are examples of tetrahedra with their largest dihedral angles not being opposite to their longest edges, see [\[54\]](#) for more 3D effects not occurring in 2D at all), there appear many bifurcation branches during refinement processes, etc. However, the current capacity of modern computers allow us, in principle, to check mesh regularity properties for a large number of (initial) tetrahedral shapes, and make some (positive or negative) conclusion on regularity based on this, which is often called proving *numerical regularity* in a positive case.

Many 3D numerical tests performed clearly demonstrate that the LE bisection seems to produce regular families of tetrahedral partitions. For example, in [\[54\]](#), the following initial tetrahedron shapes are selected for extensive tests: path tetrahedron, Sommerville space filler, cube corner, regular tetrahedron, needle, and the results show that the ratio $\frac{\text{meas}_3 S}{(\text{diam} S)^3}$ seems to be bounded from below by a positive constant (cf. [Definition 4](#)) for all generated subtetrahedra.

In [\[12\]](#) Rivara and Levin considered a classical three-dimensional longest-edge refinement method. Empirical experimentation was provided showing that the solid angle decreases slowly with the refinement iteration, and that a quality-element improvement behaviour holds in practice, similarly to the two-dimensional case. However, there have not been given mathematical results guaranteeing the non-degeneracy property of the resulting tetrahedral meshes, although many experiments suggest this property holds.

3.4. The other n -section-like algorithms

Several another variants of the bisection-type algorithms suitable for finite elements were also proposed, analysed and numerically tested in [\[57–62,55,63,64\]](#) (see also references therein).

In two dimensions, the *four-triangles longest-edge partition* (4T-LE) (see e.g. [\[11,51\]](#)) bisects a triangle into four new triangles as follows: the original triangle is first bisected by its longest edge and then the two remaining triangles are bisected by joining the new midpoint of the longest edge to the midpoints of the two (remaining) edges of the original triangle. An alternative scheme is to connect the midpoints of the edges by lines parallel to the edges. This again yields four subtriangles, each being similar to the original parent triangle and therefore inheriting its shape quality (good or bad). This

latter subdivision scheme is referred to as the ‘natural’ or self-similar (SS) pattern. As a hybrid variant consists in using 4T-LE and 4T-SS subdivision deployed independently on triangles where one or other scheme may yield better quality triangle shapes, see [65].

A recursive approach is proposed by Kossaczky [66]. This algorithm imposes certain restrictions and preprocessing in the initial mesh. The 3D algorithm is equivalent to that given in [58]. Maubach [67] develops an algorithm for d -simplicial meshes generated by reflection. Although the algorithm is valid in any dimension and the number of similarity classes is bounded, it cannot be applied for a general tetrahedral mesh. An additional closure refinement is needed to avoid incompatibilities. Also Mukherjee [57] has presented an algorithm equivalent to [58,60], and proves the equivalence with [67].

Less attention has been given to LE subdivisions based on the insertion of two, three, ... equidistant points on the longest-edge of triangle or tetrahedra, although their application seems to be rather natural when for instance narrowed or skinny elements are permitted as in the case of simulation on sharpened geometries domains. It can be argued that for example in adapted anisotropic meshes, the skinny triangles or tetrahedra are permitted, for example in highly anisotropic solutions near the boundaries of the domains like those happening in the CFD applications.

4. Some open problems

Here we list some open problems in and around the topic of the paper.

1. Prove that the LE bisection algorithms produce regular families of tetrahedral partitions. Also, prove this result (or demonstrate some degeneracy effects) in dimensions higher than two, and also for values $n > 2$.
2. Derive estimates for the number of similarity classes of simplices produced by the LE n -section algorithms. Some results in that direction for a few particular tetrahedral shapes are reported in [54,56,68].
3. Analyse the self-improvement of the quality of partitions for dimensions higher than 2 and also for values $n > 2$. In the two-dimensional case and for $n = 2$, the self-improvement property has been showed for LE bisection algorithms in [51,11] and for some other variants of these algorithms—in [61,69].
4. Develop suitable mesh movement or removal post-processing techniques to eliminate bad quality elements during LE n -section refinement processes.
5. Design some efficient algorithms and data structures for real-time applications of the techniques proposed.
6. Is there a parallelization algorithm independent of the subdivision method to afford large meshes operation?
7. Use of space-filling curves for domain automatic decomposition of tetrahedral meshes in connected subdomains. Some preliminary examples are given in [23].

In addition, the readers may consult the work by Bern and Eppstein [50] where many open problems of similar types appearing in a close field of discrete and computation geometry are listed.

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