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A note on "Some inequalities in inner product spaces related to the generalized triangle inequality" by S.S. Dragomir et al.

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ABSTRACT

In this note, we present an affirmative answer to a question presented in the paper "Some inequalities in inner product spaces related to the generalized triangle inequality" by S.S. Dragomir et al. [Appl. Math. Comput. 217 (18) (2011) 7462–7468].

In a recent paper [1], Dragomir et al. proposed the open problem of whether constant $\frac{1}{2}$, appearing in the following two theorems, is the best possible. In this short note, we answer in affirmative sense this question.

Theorem 1 [1, Theorem 6]. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in H$, for all $i \in \{1,\ldots,n\}$ and $p_i \geqslant 0$ with $\sum_{i=1}^n p_i = 1$. Suppose there exist constants $r_i > 0$, $i \in \{1,\ldots,n\}$, so that*

$$\left\| x_i - \sum_{j=1}^n p_j x_j \right\| \leqslant r_i$$

for all $i \in \{1,\ldots,n\}$. Then

$$(0 \leqslant) \sum_{i=1}^n p_i \|x_i\| - \left\| \sum_{i=1}^n p_i x_i \right\| \leqslant \frac{1}{2} \cdot \frac{\sum_{i=1}^n p_i r_i^2}{\|\sum_{i=1}^n p_i x_i\|} \tag{1}$$

provided that $\sum_{i=1}^n p_i x_i \neq 0$. □

Theorem 2 [1, Theorem 7]. *Let x_i, p_i and r_i be as in the statement of previous theorem. Then*

$$0 \leqslant \left(\sum_{i=1}^n p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^n p_i x_i \right\| \leqslant \frac{1}{2} \cdot \frac{\sum_{i=1}^n p_i r_i^2}{\|\sum_{i=1}^n p_i x_i\|}. \qquad \square \tag{2}$$

Our example is the following.

Let us consider the Euclidean space $\mathbb{R}^2, x_1 = (1,0)$ and $x_2 = (\alpha,\beta)$ with $\|x_i\|^2 = 1$ for $i = 1, 2$, that is $\alpha^2 + \beta^2 = 1$. We choose $p_1 = \frac{1}{4}$ and $p_2 = \frac{3}{4}$. Then $p_1\|x_1\| + p_2\|x_2\| = 1$ and

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$$p_1x_1 + p_2x_2 = \left(\frac{1}{4}+\frac{3}{4}\alpha, \frac{3}{4}\beta\right),$$

$$\|x_1 - (p_1x_1 + p_2x_2)\|^2 = \left(\frac{3}{4}-\frac{3}{4}\alpha\right)^2 + \left(\frac{3}{4}\beta\right)^2 = \frac{18}{16}(1-\alpha) = r_1^2,$$

$$\|x_2 - (p_1x_1 + p_2x_2)\|^2 = \left(\frac{-1}{4}+\frac{1}{4}\alpha\right)^2 + \left(\frac{1}{4}\beta\right)^2 = \frac{2}{16}(1-\alpha) = r_2^2,$$

$$\|p_1x_1 + p_2x_2\|^2 = \left(\frac{1}{4}+\frac{3}{4}\alpha\right)^2 + \left(\frac{3}{4}\beta\right)^2 = \frac{10}{16}+\frac{6}{16}\alpha.$$

In this case, inequalities (1) and (2) are:

$$1 - \frac{1}{4}\sqrt{10+6\alpha} \leqslant \frac{1}{2}\cdot\frac{3(1-\alpha)}{2\sqrt{10+6\alpha}}$$

or, equivalently,

$$\frac{1-\frac{1}{4}\sqrt{10+6\alpha}}{\frac{3(1-\alpha)}{2\sqrt{10+6\alpha}}} \leqslant \frac{1}{2}. \tag{3}$$

In order to prove that the previous inequality is sharp, assume that there exists a constant $c > 0$ such that inequality (3) holds with c, i.e.

$$\frac{1-\frac{1}{4}\sqrt{10+6\alpha}}{\frac{3(1-\alpha)}{2\sqrt{10+6\alpha}}} \leqslant c. \tag{4}$$

Now, since $\alpha \in (0,1)$ we may consider $\alpha = 1 - \varepsilon$. Letting $\varepsilon \to 0$ and taking into account L'Hopital rule we get:

$$\lim_{\varepsilon\to 0}\frac{1-\frac{1}{4}\sqrt{16-6\varepsilon}}{\frac{3\varepsilon}{2\sqrt{16-6\varepsilon}}} = \lim_{\varepsilon\to 0}\frac{8}{3\varepsilon}\left(1-\frac{1}{4}\sqrt{16-6\varepsilon}\right) = \frac{1}{2}.$$

Therefore, constant $\frac{1}{2}$ is the best possible in Theorems 1 and 2. □

References

[1] S.S. Dragomir, Y.J. Cho, S.S. Kim, Some inequalities in inner product spaces related to generalized triangle inequality, Appl. Math. Comput. 217 (18) (2011) 7462–7468.