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Two-sided estimation of diameters reduction rate for the longest edge n-section of triangles with $n \geqslant 4$

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ABSTRACT

In this work we study the diameters reduction rate for the iterative application of the longest edge (LE) n-section of triangles for $n \geqslant 4$. The maximum diameter d_k^n of all triangles generated at the kth iteration of the LE n-section is closely connected with the properties of the triangular mesh generated by this refinement scheme. The upper and the lower bounds for d_k^2 were proved by Kearfott in [9] and for d_k^3 by Plaza et al. [12]. Here, we derive the two-sided estimates for d_k^n with $n \geqslant 4$.

1. Introduction

Many applications in science, engineering, statistics, and mathematics require to decompose a given domain into small subdomains called elements. The longest edge bisection algorithm is one of the most popular refinement techniques since it is very simple, computationally cheap and can be easily applied in higher dimensions. Let T be a triangle in $\mathbb{R}^2$. We find the longest edge (LE) of T, insert $n - 1$ equally-spaced points in the LE and connect them by line segments to the opposite vertex. This yields the generation of n new sub-triangles whose parent is T. Now, continue this process iteratively. In particular, nested sequences of meshes so generated, where each element of the sequence is a member of an element in its predecessor mesh are of interest in several fields of computational applied mathematics.

The longest edge bisection algorithm was originally designed for solving nonlinear equations [7,16]. Lastly mainly due to effort of Rivara in a series of papers that began in 1984 [14], longest edge methods started to become popular in the practical use of the finite element method.

For better understanding of repeated refinement of triangles, the longest edge (diameter) of successive triangle generation has been studied. For the LE 2-section, also the LE bisection, these studies began with the paper of Kearfott [9] who proved a bound on the behavior of diameter length of any triangle obtained. Later Stynes [18] presented a more sharp bound for certain initial triangles. After that, Stynes [19] and Adler [1] improved this bound for all triangles.

In relation to the case of $n = 3$, in [12] a mathematical proof of how fast the diameters of a triangular mesh tend to zero after repeated trisection is presented. However this study has not been carried out yet for LE n-section, $n \geqslant 4$. Also, it should be noted that these series of works have not been fully extended to higher dimension than two, for tetrahedra and m-simplices. Horst [8] partially studied the bisection of regular simplices and reproved a bound of the convergence rate in terms of diameter reduction firstly introduced by Kearfott in [9].

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The refinement schemes used in the works of Kearfott, Stynes, Adler, Horst [9,19,1,8] and herein do not necessarily get conformity meshes (the intersection of non-disjoint triangles is either a common vertex or a common edge). It should be noted that this condition is not a problem for the convergence rate study. Other methods not based on the LE, for example the generalized conforming bisection algorithm [5], also have proved to fulfill the convergence of diameters.

In particular the LE n-section methods in two dimensions for $n \geqslant 4$ have been recently studied: in [20] it is proved, and reproved in [11] using a shorter manner, that the LE n-section refinement scheme produces a sequence of triangular meshes such that the minimum interior angle of the triangles converges to zero when $n \geqslant 4$. This comes as a complement of earlier results which showed that the non-degeneracy property (the minimum angle is bounded away from zero) holds in the cases of the LE bisection [15] and the LE trisection [13] methods.

In this paper we give upper and lower bounds for the convergence rate in terms of diameter reduction in the iterative LE n-section of triangles for $n \geqslant 4$. Similar studies have been carried out in the past for $n = 2, 3$ [9,19,1,8,12] and now for the case of $n \geqslant 4$ in this paper. Numerical tests presented in this paper with the 4-section method reveal that for some triangles, the percentage of better triangles appearing during the refinement is superior to the percentage of poorer triangles generated. However, an improved empirical work treating the numerical behavior of the LE n-section seems necessary.

2. The upper bound for the diameters in the LE n-section ($n \geqslant 4$)

In the following theorem we give a sharpened upper bound for the repeated LE n-section ($n \geqslant 4$) of triangles:

Theorem 1. *Let d_k^n be the maximum diameter of triangles generated at kth iteration of the LE n-section ($n \geqslant 4$) of $\triangle ABC$ with $|\overline{AB}| \leqslant |\overline{CA}| \leqslant |\overline{BC}|$. Then for $k \geqslant 1$ the following inequality holds:*

$$d_k^n \leqslant \left(\frac{\sqrt{n^2 - n + 1}}{n}\right)^{k-1} |\overline{CA}|. \tag{1}$$

It should be noted that we take d_0^n as the longest edge of $\triangle ABC$, i.e. $d_0^n = |\overline{BC}|$.

Before, we give some previous lemmas which are used in the proof:

Lemma 1. *(Theorem of Stewart, [17]) Let $\overline{AS}$ a cevian from A in $\triangle ABC$. Then:*

$$|\overline{AS}|^2 |\overline{BC}| = |\overline{AB}|^2 |\overline{SC}| + |\overline{CA}|^2 |\overline{BS}| - |\overline{BS}||\overline{SC}||\overline{BC}|. \tag{2}$$

Lemma 2. *Let S be a point in the side $\overline{BC}$ of $\triangle ABC$ with $|\overline{AB}| \leqslant |\overline{CA}| \leqslant |\overline{BC}|$, H be the foot of the height from A, X and Y points in segment $\overline{BC}$ such that $|\overline{BX}| = |\overline{CY}|$ and $\overline{BX} \cap \overline{CY} = \phi$ i.e. have an empty intersection. Then the following assertions hold.*

(i) $|\overline{AS}| \leqslant |\overline{CA}|$, see Fig. 1(a).
(ii) $|\overline{BH}| \leqslant |\overline{HC}|$, see Fig. 1(b).
(iii) $|\overline{AX}| \leqslant |\overline{AY}|$, see Fig. 1(b) and (c).

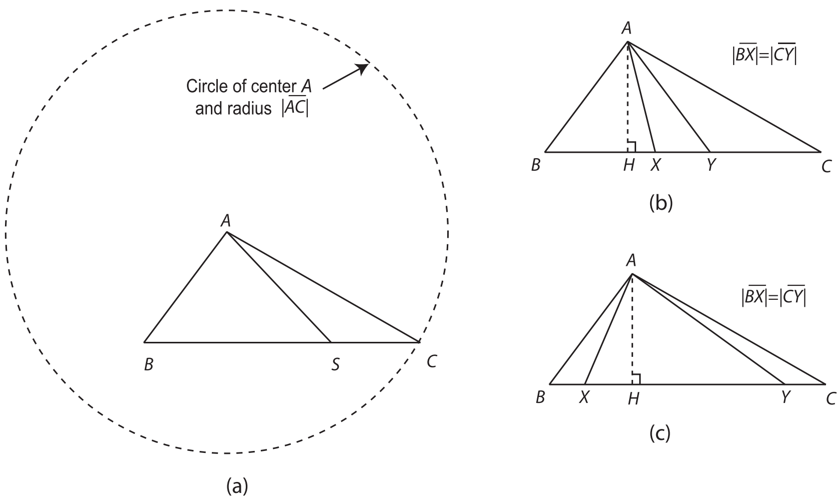

Fig. 1. (a) $|\overline{AS}| \leqslant |\overline{CA}|$ for each point $S \in \overline{BC}$, (b) $X \in \overline{HC}, Y \in \overline{HC}$, (c) $X \in \overline{BH}, Y \in \overline{HC}$.

Remark. The equality in (i) holds for $S = C$. The equality in (ii) and (iii) holds for $|\overline{AB}| = |\overline{CA}|$.

Proof.

(i) Note that B belongs to the closed ball with center A and radius $|\overline{CA}|$ as $|\overline{AB}| \leqslant |\overline{CA}|$. Using the fact that any closed ball in the plane is a convex set, each point of the segment $\overline{BC}$ belongs to this ball. Hence $|\overline{AS}| \leqslant |\overline{CA}|$.
(ii) By the Pithagoras theorem we have

$$|\overline{CA}|^2 - |\overline{HC}|^2 = |\overline{AH}|^2 = |\overline{AB}|^2 - |\overline{BH}|^2,$$

thus

$$|\overline{CA}|^2 - |\overline{AB}|^2 = |\overline{HC}|^2 - |\overline{BH}|^2.$$

Now, taking into account the fact that $|\overline{AB}| \leqslant |\overline{CA}|$, we have $|\overline{BH}| \leqslant |\overline{HC}|$.
(iii) We proceed to prove that $|\overline{HX}| \leqslant |\overline{HY}|$. We have two possibles cases about the location of X and Y:

Case 1: $X \in \overline{HC}, Y \in \overline{HC}$, see Fig. 1 (b). We have $|\overline{HX}| = |\overline{HY}| - |\overline{XY}|$, thus $|\overline{HX}| \leqslant |\overline{HY}|$.
Case 2: $X \in \overline{BH}, Y \in \overline{HC}$, see Fig. 1 (c). We have

$$|\overline{HX}| = |\overline{BH}| - |\overline{BX}| \leqslant |\overline{HC}| - |\overline{BX}| = |\overline{HC}| - |\overline{CY}| = |\overline{HY}|.$$

Obviously the case $X \in \overline{BH}, Y \in \overline{BH}$ is impossible. Then $|\overline{HX}| \leqslant |\overline{HY}|$.
By the Pithagoras theorem, $|\overline{AX}|^2 - |\overline{HX}|^2 = |\overline{AH}|^2 = |\overline{AY}|^2 - |\overline{HY}|^2$, hence

$$|\overline{HY}|^2 - |\overline{HX}|^2 = |\overline{AY}|^2 - |\overline{AX}|^2.$$

From the above equality and the inequality $|\overline{HX}| \leqslant |\overline{HY}|$ we conclude that $|\overline{AX}| \leqslant |\overline{AY}|$.

Lemma 3. *Let $\triangle ABC$ such that $|\overline{AB}| \leqslant |\overline{CA}| \leqslant |\overline{BC}|$ and $X_1, X_2, \ldots, X_{n-1}$, $n \geqslant 2$, points of $\overline{BC}$ such that $|\overline{BX_1}| = |\overline{X_1X_2}| = \cdots = |\overline{X_{n-1}C}| = \frac{1}{n}|\overline{BC}|$. Then the following assertions hold:* (i) For $n \geqslant 4$ the medium edge length of each triangle $\triangle BAX_1, \triangle X_1AX_2, \ldots, \triangle X_{n-1}CA$ is less or equal than $|\overline{AX_{n-1}}|$. (ii) $|\overline{AX_{n-1}}| \leqslant \frac{\sqrt{n^2-n+1}}{n}|\overline{CA}|$, where the medium edge of an arbitrary triangle $\triangle UVW$ such that $|\overline{UV}| \leqslant |\overline{UW}| \leqslant |\overline{VW}|$ is $\overline{UW}$.

Proof.

(i) From Lemma 2 (ii) applied to $\triangle ABC$ we have that $|\overline{AX_1}| \leqslant |\overline{AX_{n-1}}|$. From Lemma 2 (i) applied to a $\triangle AX_1X_{n-1}$ we have $|\overline{AX_i}| \leqslant |\overline{AX_{n-1}}|, i \in \{1, 2, \ldots n-1\}$. Furthermore, by Lemma 2 (ii) we deduce that $X_{n-2} \in \overline{HC}$, being H the foot of the height from A, see Fig. 2. In fact, the condition $n \geqslant 4$ implies that $n-2 \geqslant 2$ and consequently both points X_{n-1} and X_{n-2} are on the segment $\overline{MC}$, where M is the midpoint of $\overline{BC}$. Now the fact that $|\overline{BH}| \leqslant |\overline{HC}|$ implies that $X_{n-2} \in \overline{HC}$. Hence $\angle AX_{n-2}X_{n-1} \geqslant \frac{\pi}{2}$, and so $|\overline{BX_1}| = |\overline{X_1X_2}| = \cdots = |\overline{X_{n-1}C}| < |\overline{AX_{n-1}}|$, see Fig. 2.
If the medium edge of $\triangle BAX_1$ is $\overline{AX_1}$ or $\overline{BX_1}$ then the proof is completed, as we have proved before $|\overline{AX_1}| \leqslant |\overline{AX_{n-1}}|$ and $|\overline{BX_1}| \leqslant |\overline{AX_{n-1}}|$. Suppose that $|\overline{AB}|$ is the length of medium edge of $\triangle BAX_1$, then the length of its longest edge is, either $|\overline{AX_1}|$ or $|\overline{BX_1}|$. Note that last two segments, are less or equal than $|\overline{AX_{n-1}}|$, as previously proven.
(ii) Note that $|\overline{BX_{n-1}}| = \frac{n-1}{n}|\overline{BC}|$ and $|\overline{CX_{n-1}}| = \frac{1}{n}|\overline{BC}|$. By Lemma 1:

$$|\overline{AX_{n-1}}|^2|\overline{BC}| = |\overline{AB}|^2 \cdot \frac{1}{n}|\overline{BC}| + |\overline{CA}|^2 \cdot \frac{n-1}{n}|\overline{BC}| - \left(\frac{n-1}{n^2}\right)|\overline{BC}|^3,$$

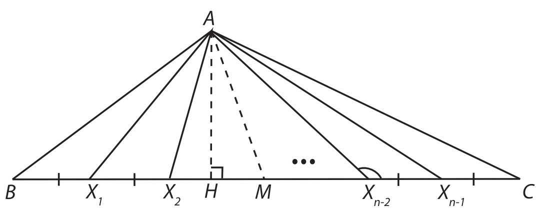

Fig. 2. $|\overline{BX_1}| = |\overline{X_1X_2}| = \cdots = |\overline{X_{n-1}C}| < |\overline{AX_{n-1}}|$.

thus

$$|\overline{AX_{n-1}}|^2 = \frac{|\overline{AB}|^2 + (n-1)|\overline{CA}|^2}{n} - \left(\frac{n-1}{n^2}\right)|\overline{BC}|^2.$$

Taking into account that $|\overline{AB}| \leqslant |\overline{CA}| \leqslant |\overline{BC}|$, we have:

$$|\overline{AX_{n-1}}|^2 \leqslant \frac{|\overline{CA}|^2(n-1) + |\overline{CA}|^2}{n} - \left(\frac{n-1}{n^2}\right)|\overline{CA}|^2 = \left(\frac{n^2-n+1}{n^2}\right)|\overline{CA}|^2$$

from where we have the inequality which proves the result of the Lemma. □

At this point, we follow with the proof of the main result, Theorem 1.

Proof of Theorem 1: Let us consider the sequence $\{I_k\}_{k=1}^{\infty}$ such that I_k is the maximum medium edge of all triangles obtained at kth iteration. Note that at iteration $k+1$, each longest edge previously obtained at kth iteration is subdivided into n equal parts. Each of these parts is the shortest edge of at least one of the triangles obtained at $(k+1)$th iteration. Then $d_{k+1}^n = I_k$. Using Lemma 3 (i) and (ii) we have $I_{k+1} \leqslant \frac{\sqrt{n^2-n+1}}{n} I_k$. Consequently, $I_k \leqslant \left(\frac{\sqrt{n^2-n+1}}{n}\right)^k I_0$. Thus, we follow that $d_k^n \leqslant \left(\frac{\sqrt{n^2-n+1}}{n}\right)^{k-1} d_1^n$. Obviously $d_1^n = I_0 = |\overline{CA}|$, thus $d_k^n \leqslant \left(\frac{\sqrt{n^2-n+1}}{n}\right)^{k-1} |\overline{CA}|$. □

3. The lower bound for diameters in the LE n-section ($n \geqslant 4$)

We now provide a lower bound for the LE n-section of triangles.

Theorem 2. *Let d_k^n be the maximum diameter of all triangles obtained at kth iteration of the LE n-section ($n \geqslant 4$) to a given arbitrary triangle ABC with edges $a = |\overline{BC}|, b = |\overline{CA}|$ and $c = |\overline{AB}|$, such that $c \leqslant b \leqslant a$. Then, there exists constants p, q, r, s, t and u only dependent on a, b, c and n, such that the following assertions hold:*

(i) For $n = 4, d_k^n \geqslant \left(\frac{1}{2}\right)^k \sqrt{pk^2 + qk + r}$.

(ii) For $n \geqslant 5, d_k^n \geqslant \sqrt{s\frac{1}{n^k} + t\left(\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k + u\left(\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k}$.

Proof. Let us consider the triangle sequence $\{\Delta_k\}_{k=0}^{\infty}$ such that $\Delta_0 = \triangle A_0B_0C_0, A_0 = A, B_0 = B, C_0 = C$, and for each $k \geqslant 0$ let $\Delta_{k+1} = \triangle A_{k+1}B_{k+1}C_{k+1}$ where $A_{k+1} \in \overline{B_kC_k}$ such that $|\overline{C_kA_{k+1}}| = \frac{1}{n}|\overline{B_kC_k}|, B_{k+1} = C_k$ y $C_{k+1} = A_k$, see Fig. 3. It can be noted that for each $k \geqslant 1, |\overline{A_kB_k}| \leqslant |\overline{C_kA_k}| \leqslant |\overline{B_kC_k}|$; from where we have for each $k \geqslant 0, \Delta_{k+1}$ is one of the n triangles obtained by applying the LE n-section to triangle Δ_k.

Let us now consider the sequence $\{a_k\}_{k=0}^{\infty}$ where $a_k = |\overline{B_kC_k}|$. Using Lemma 1 applied to $\triangle A_{k+1}B_{k+1}C_{k+1}$ having cevian $|\overline{A_{k+1}A_{k+2}}|$ following recurrence equation is obtained:

$$a_{k+3}^2 - \frac{n-1}{n}a_{k+2}^2 + \frac{n-1}{n^2}a_{k+1}^2 - \frac{1}{n^3}a_k^2 = 0,$$

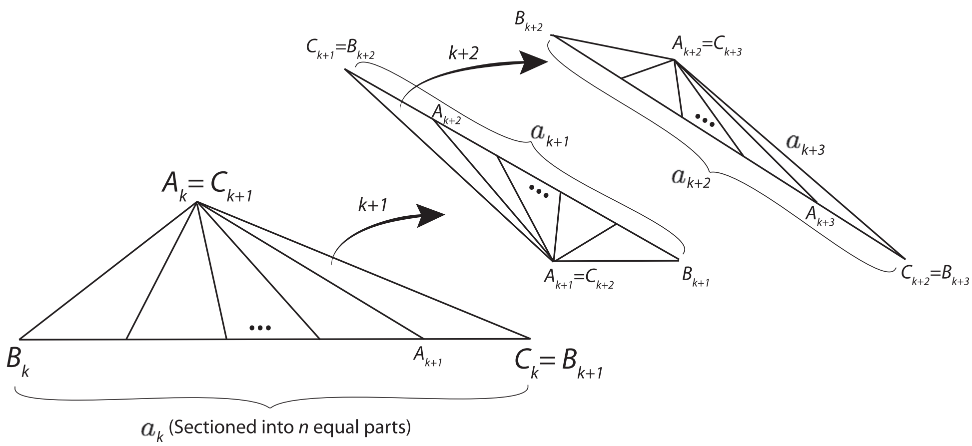

Fig. 3. On generation of triangle $\Delta_k = \triangle A_kB_kC_k, \Delta_{k+1} = \triangle A_{k+1}B_{k+1}C_{k+1}$ and $\Delta_{k+2} = \triangle A_{k+2}B_{k+2}C_{k+2}$.

where $a_0 = a, a_1 = b$ and $a_2 = \frac{\sqrt{nc^2+n(n-1)b^2-(n-1)a^2}}{n}$. Take $y_k = a_k^2$, then:

$$y_{k+3} - \frac{n-1}{n} y_{k+2} + \frac{n-1}{n^2} y_{k+1} - \frac{1}{n^3} y_k = 0,$$

where $y_0 = a^2, y_1 = b^2$ and $y_2 = \frac{nc^2+n(n-1)b^2-(n-1)a^2}{n^2}$. It can be noted from the construction of sequence $\{a_k\}_{k=0}^{\infty}$ that each term of the sequence is positive.

The characteristic equation is:

$$\lambda^3 - \frac{n-1}{n}\lambda^2 + \frac{n-1}{n^2}\lambda - \frac{1}{n^3} = 0.$$

At this point, two separated situations can be given: (i) $n = 4$, where a root of multiplicity 3 is obtained, and (ii) $n \geqslant 5$ where three real roots are obtained.

(i) Case $n = 4$. The solution of the characteristic equation is $\lambda = \frac{1}{4}$, of multiplicity 3, and then:

$$y_k = \left(\frac{1}{4}\right)^k (pk^2 + qk + r),$$

where p, q y r are real constants only dependent on a, b y c and are determined from the system obtained for the initial conditions:

$$\left.\begin{aligned} r &= a^2 \\ \tfrac{1}{4}(p+q+r) &= b^2 \\ \tfrac{1}{16}(4p+2q+r) &= \tfrac{4c^2+12b^2-3a^2}{16} \end{aligned}\right\}$$

from where $p = -a^2 + 2b^2 + 2c^2, q = 2b^2 - 2c^2, r = a^2$. Note that $d_k^n \geqslant a_k$ and then $d_k^n \geqslant \sqrt{y_k} = (\frac{1}{2})^k \sqrt{pk^2 + qk + r}$.

(ii) Case $n \geqslant 5$. The characteristic equation has three real roots: $\lambda_1 = \frac{1}{n}, \lambda_2 = \frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}$ and $\lambda_3 = \frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}$. Thus:

$$y_k = \left(\frac{1}{n}\right)^k s + \left(\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k t + \left(\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k u.$$

where s, t and u are real constants only dependent on a, b, c and n.

Such constants are determined from the system obtained for the initial conditions:

$$\left.\begin{aligned} s + t + u &= \alpha \\ \lambda_1 s + \lambda_2 t + \lambda_3 u &= \beta \\ \lambda_1^2 s + \lambda_2^2 t + \lambda_3^2 u &= \gamma \end{aligned}\right\}$$

where $\alpha = a^2, \beta = b^2$ and $\gamma = \frac{nc^2+n(n-1)b^2-(n-1)a^2}{n^2}$.

We obtain:

$$s = \frac{\beta(\lambda_2+\lambda_3)-\alpha\lambda_2\lambda_3-\gamma}{-(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$
$$t = \frac{\beta(\lambda_1+\lambda_3)-\alpha\lambda_1\lambda_3-\gamma}{-(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)},$$
$$u = \frac{\beta(\lambda_1+\lambda_2)-\alpha\lambda_1\lambda_2-\gamma}{-(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

It should be noted that $d_k^n \geqslant a_k$, and so: $d_k^n \geqslant \sqrt{(\frac{1}{n})^k s + \left(\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k t + \left(\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k u}$. Note that $a_k = \sqrt{(\frac{1}{n})^k s + \left(\frac{n^2-2n+n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k t + \left(\frac{n^2-2n-n^{\frac{3}{2}}\sqrt{n-4}}{2n^2}\right)^k u} > 0$ for each $k \geqslant 0$ as $a_k = |\overline{B_k C_k}|$, thereby we are assuring that the lower bound is positive. □

4. Numerical tests

We study the convergence rate of the repeated LE 4-section where the refinement scheme consists in the quatersection of triangles by their longest edge. Figs. 4(a), (b) and (c) graph the upper bound, the exact maximum diameters and the lower bound for the three initial triangles with diameters equal 1. The diameters rate is studied through an iterative refinement of 35 levels. The (x, y) coordinates for the initial three triangles are:

$$\begin{aligned} \Delta 1 &= (0,0) \quad (0.5, \sqrt{3}/2) \quad (1,0) \\ \Delta 2 &= (0,0) \quad (0.1, 0.1) \quad (1,0) \\ \Delta 3 &= (0,0) \quad (0.4, 0.01) \quad (1,0) \end{aligned}$$

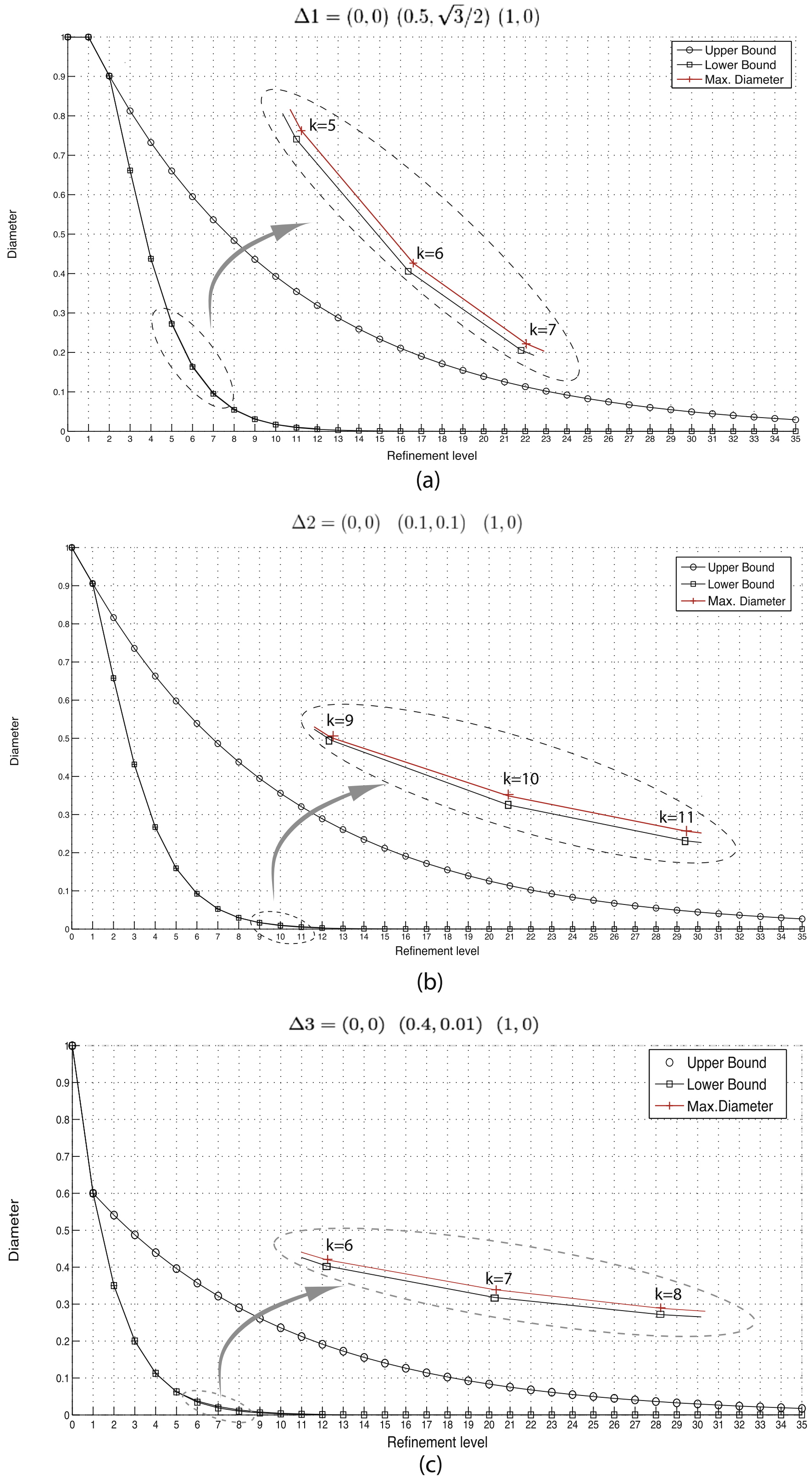

Fig. 4. The upper and lower bounds and the maximum diameters for the LE 4-section. (a) Triangle test Δ_1, (b) Triangle test Δ_2 and (c) Triangle test Δ_3.

Table 1
Values for lower bound, exact diameters and upper bound for triangles Δ_1, Δ_2 and Δ_3 in twelve refinement levels.

k	Δ_1			Δ_2			Δ_3		
	lower	diam.	upper	lower	diam.	upper	lower	diam.	upper
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	1.0000	1.0000	1.0000	0.9055	0.9055	0.9055	0.6001	0.6001	0.6001
2	0.9014	0.9014	0.9014	0.6576	0.6576	0.8162	0.3501	0.3501	0.5409
3	0.6614	0.6614	0.8125	0.4316	0.4316	0.7358	0.2001	0.2001	0.4876
4	0.4375	0.4375	0.7324	0.2672	0.2672	0.6632	0.1126	0.1126	0.4395
5	0.2724	0.2747	0.6602	0.1593	0.1593	0.5978	0.0626	0.0626	0.3961
6	0.1631	0.1654	0.5951	0.0925	0.0925	0.5388	0.0344	0.0375	0.3571
7	0.0950	0.0967	0.5364	0.0527	0.0527	0.4857	0.0188	0.0227	0.3219
8	0.0543	0.0554	0.4835	0.0296	0.0296	0.4378	0.0102	0.0133	0.2901
9	0.0305	0.0312	0.4358	0.0164	0.0167	0.3946	0.0055	0.0078	0.2615
10	0.0169	0.0174	0.3928	0.0090	0.0100	0.3557	0.0029	0.0045	0.2357
11	0.0093	0.0103	0.3541	0.0049	0.0060	0.3206	0.0016	0.0026	0.2125
12	0.0051	0.0062	0.3192	0.0027	0.0035	0.2890	0.0008	0.0015	0.1915

It should be noted in those figures that the maximum diameters curve is quite close to the lower bound curve. More specifically, the values of exact diameters coincide with the lower bound values when the refinement level goes from $k = 1$ to $k = 4, k = 8$ and $k = 5$ respectively for triangles Δ_1, Δ_2 and Δ_3. This event has been indicated in Figs. 4 (a), (b) and (c) by zooming the curves from those k values forward.

To clearly note the goodness of the bounds, Table 1 gives the numerical values for twelve levels of refinements ($k = 12$) of the exact diameters and the lower and upper bounds for the triangles Δ_1, Δ_2 and Δ_3.

It is worth to note that a complete discussion of the applicability of the LE n-section methods has not been carried out yet. The angles degeneracy showed in the works [11,20] is stated under the assumption of an asymptotic premise and however an improved empirical work treating numerical behavior seems necessary.

With the aim to give some evidence of the behavior of the LE 4-section method we conduct now a numerical study concentrated on angles and shape quality, where following variables are inspected through five iterative applications of the LE 4-section:

- Average of minimum angles (AvMin)
- Average of maximum angles (AvMax)
- Minimum angles (Min)
- Maximum angles (Max)
- Triangles (%) with minimum angle less than 15° (%*Min*15°)
- Triangles (%) with maximum angle greater than 150° (%*Max*150°)
- Triangles (%) with minimum angle greater than 15° and maximum angle less than 150° (%[15°150°])
- Average of quality measure $Q(\Delta) = \frac{4 \cdot a \cdot \sqrt{3}}{\sum_{i=1}^{3} l_i^2}$ where l_i is the length of the ith edge and a the area of the triangle Δ. If for an arbitrary triangle Δ, $Q(\Delta)$ equals 1 then Δ is regular, while if $Q(\Delta)$ equals 0 it is indicating that Δ is a degenerated triangle. (*AvQ*).

In Tables 2 and 3 it is presented the angles and quality results for triangles Δ_2 and Δ_3. It is worth noting that although in general minimum angles (*Min*) deteriorate for some new emerging triangles, however it is observed an improvement in average (*AvMin*) from the initial triangle (6.340°) to the final mesh (20.717°) at level 5 with $4^5 = 1024$ triangles. Also maximum angles improve in average, see *AvMax* in the 2^{th} column. In the 5^{th} column it is showed the percentage of triangles with minimum angle less than 15° and in the 6^{th} column with maximum angles greater than 150°. Triangles with small angles diminish for Δ_2 and Δ_3, indicating some improvement with respect to minimum angle. Triangles with big angles also diminish for Δ_3 while for Δ_2 a percentage of 20.996% triangles remain at level 5.

It can be also remarked that triangles with minimum angle greater than 15° and maximum angle less than 150° (*%[15°150°]*) increases as the refinement increase, see 7^{th} in Table 2. This is, from triangle Δ_2 that is of poor quality, one gets up to 62.304% of new triangles with considerably better quality at the fifth refinement level. In agreement with that, the average of the quality measure (*AvQ*) also improves, from 0.188 to 0.486. An analogous behavior can be seen for triangle Δ_3, see Table 3. Hence, it is clear that the percentage of better triangles is superior to the percentage of poorer triangles when the refinement levels increase.

In view of that, it seems an interesting open problem to study the LE 4-Section in combination with some mesh improvement technique as node relaxation, Delaunay remeshing, local modifications as flipping edges etc. [3,4,10]. To this end one may apply improvement techniques to those poorer triangles appearing in the refinement process, thereby obtaining a superior mesh quality. However, these aspects of mesh improvement are not the objective of the present work.

In relation to the regular type triangles, as for example Δ_1, it should mentioned that whatever LE n-section scheme yields in some extents a deterioration in the quality of new triangles. Whether this deterioration may approaches a degeneration or

Table 2
Study of angles and quality on the LE 4-section for triangle Δ_2.

AvMin	*AvMax*	*Min*	*Max*	%*Min*15°	%*Max*150°	%[15°150°]	AvQ
6.340	128.659	6.340	128.659	100	0	0	0.188
13.855	146.209	2.406	171.253	75	50	25	0.319
16.304	132.864	1.261	176.332	50	37.500	50	0.386
18.962	129.263	0.776	177.961	46.875	26.562	53.125	0.440
19.956	124.358	0.525	178.698	39.062	27.734	60.937	0.469
20.717	122.616	0.379	179.094	37.695	20.996	62.304	0.486

Table 3
Study of angles and quality on the LE 4-section for triangle Δ_3.

AvMin	*AvMax*	*Min*	*Max*	%*Min*15°	%*Max*150°	%[15°150°]	*AvQ*
0.954	177.613	0.954	177.613	100	100	0	0.022
1.891	174.828	0.681	178.363	100	100	0	0.046
3.791	167.599	0.511	178.807	100	87.500	0	0.094
7.238	156.082	0.397	179.091	87.500	76.562	12.500	0.174
10.828	147.683	0.317	179.285	76.953	58.984	23.046	0.254
13.725	139.887	0.238	179.422	66.015	53.515	33.984	0.319

not depend on n. For that reason in the case of regular triangles, the most appropriate subdivision scheme might not be a LE scheme. For example, one may use an scheme where the parent triangle is subdivided into congruent subtriangles each similar to the parent triangle by tracing lines parallel to the edge and passing through the inserted points.

5. Final remarks

Proficient algorithms for mesh refinement based on the longest edge are known when $n = 2$, but less known when $n = 3$ and completely unknown when $n \geqslant 4$. The LE n-section based algorithms are surprisingly cheap and robust. They are linear in the number of elements, as the only necessary calculations are: (i) longest edges and (ii) insertion of n points in the LE sides.

In this paper we give upper and lower bounds for the convergence rate in terms of diameter reduction in the iterative LE n-section of triangles for $n \geqslant 4$. We contribute to a better understanding of the LE n-section ($n \geqslant 4$) of triangular meshes.

It should be noted that in our study we do not get mesh conformity which is necessary for example for FEM. However, if one converts a final non conforming triangle mesh to a new conforming mesh, simply by connecting those non conforming nodes with opposite vertices then the lower bound will remain the same. The upper bound for that conforming mesh will be greater than the bound obtained here.

From the properties studied in [11,20] one may infer that the practical use of the LE n-section for example in finite element simulations may be questioned. In finite element computations, either uniform or adaptive mesh refinement one may expect the use of the refinement algorithm under a compromise among the error indicator, mesh size and computational cost, [2]. However, it is worth to note that a complete discussion on that issue has not been carried out yet for the LE n-section ($n \geqslant 4$).

Numerical tests presented in this paper reveal that for some triangles, the percentage of better new triangles is superior to the percentage of poorer triangles when the refinement levels increase. This may indicates that for example 4-section is an alternative algorithm for mesh refinement. Even in the case that LE n-section may derive some element degeneration, the scheme may be of utility. To support this idea, see for example [6] where it has been shown that large angles may produce small finite element discretization error, even though the interpolation error is large.

However, an improved empirical work treating the numerical behavior of the LE n-section seems necessary. Moreover we also suggest here the application of improvement techniques as node relaxation, Delaunay remeshing, local modifications as swapping edges, etc. [3,4,10] that might improve the quality of meshes generated.

To mention just other interesting issue around the LE n-section method, it can be studied under the idea of face-to-face partition of Korotov et al. [5]. In fact, this strategy, either the LE n-section alone or just combining it with other method would appear useful even in higher dimension, for tetrahedral meshes or m-simplicial meshes. It should be noted that it is an open problem whether the LE n-section in 3D gets a degeneracy of minimum angles or not. Even this has not been formally accomplished yet neither in the LE bisection or the LE trisection in 3D.

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