"Strange as it might seem, any set, for instance R^2, can be topologically wrapped inside a unit ball."

A Triangle Inequality and its Elementary Proof

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One of the most surprising facts in a first course of calculus and analysis is that the concept of 'distance' is unambiguous in mathematics. In fact, given a set S, a distance function (or metric) d is any positive function defined over pairs of elements of S, such that, for every two elements $x, y \in S$, $d(x, y)$ is a nonnegative real number (positivity) that satisfies:

1. Reflexivity: $d(x, y) = 0$ if and only if $x = y$,
2. Symmetry: $d(x, y) = d(y, x)$,
3. The triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

A commonly proposed problem is to verify that a given function is a distance over some set. For such a problem, usually the key point is to assess whether the triangle inequality holds:

Example: Prove that if $d(x, y)$ is a distance, then

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is also a distance.

It is immediately clear that d^* satisfies positivity, reflexivity, and symmetry. The triangle inequality for d^* is a consequence of the following property:

Property: Let a, b, c be positive numbers such that $a < b + c$. Then

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

Proof: The property is proved based on a more general fact: If a,b and x are three positive numbers and $a < b$, then

$$\frac{a}{x+a} < \frac{b}{x+b}.$$

Although the proof is not difficult, a proof without words is given in Figure 1.

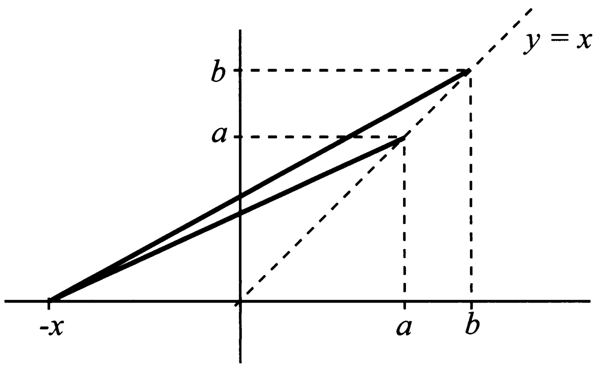

Figure 1. A visual explanation for $\frac{a}{x+a} < \frac{b}{x+b}$.

Now, for $x = 1$, and since $a < b + c < b + c + bc$, we obtain:

$$\frac{a}{1+a} < \frac{b+c}{1+b+c} < \frac{b+c+bc}{1+b+c+bc}.$$

But,

$$\frac{b+c+bc}{1+b+c+bc} < \frac{b+c+2bc}{1+b+c+bc} = \frac{b+bc+c+bc}{(1+b)(1+c)} = \frac{b}{1+b} + \frac{c}{1+c}.$$

Therefore,

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

Remark: Notice that in the same set, different distance functions may be defined. For example, the distance in this example

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

makes S a bounded set, since for every pair of points

$$0 \leq d^*(x,y) = \frac{d(x,y)}{1+d(x,y)} < 1.$$

Thus, under this metric, the distance between any two points of S is less than one. Consequently, strange as it might seem, any set, for instance R^2, can be topologically wrapped inside a unit ball. ■