

PROBLEM OF THE WEEK

Problem No. 1 (Fall 2008 Series)

Show that  $\lim_{n \rightarrow \infty} \int_0^1 \cdots \int_0^1 \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right)^2 dx_1, \dots, dx_n = \frac{1}{4}$ .

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**Solution:** by Ángel Plaza ULPGC, Spain

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Since  $(x_1 + x_2 + \cdots + x_n)^2 = \sum_{i=1}^n x_i^2 + 2 \sum_{i \neq j} x_i x_j$ , then

$$\lim_{n \rightarrow \infty} \int_0^1 \cdots \int_0^1 \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right)^2 dx_1, \dots, dx_n =$$
$$\lim_{n \rightarrow \infty} n \int_0^1 \frac{x^2}{n^2} dx + 2 \binom{n}{2} \int_0^1 \int_0^1 \frac{xy}{n^2} dxdy = \lim_{n \rightarrow \infty} \left( \frac{n}{3n^2} + 2 \frac{n(n-1)}{2} \frac{1}{4n^2} \right) =$$
$$\frac{1}{4}.$$