Solution to AMM Problem # 11441

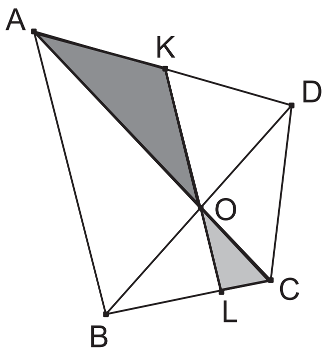
Ángel Plaza & Sergio Falcón
Department of Mathematics, Universidad de Las Palmas de Gran Canaria
35017–Las Palmas G.C. SPAIN,

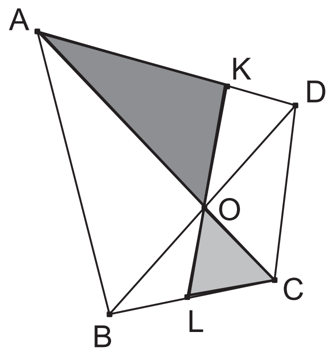
Problem# 11441 *Proposed by Y. N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan* Let $n \geq 4$, let $A_0, \ldots, A_{n-1}$ be the vertices of a convex polygon, and for each i let B_i be a point in the interior of the segment A_iA_{i+1}. (Here, and throughout, indices of points are taken modulo n.) Let C_i denote the intersection of diagonals $B_{i-2}B_i$ and $B_{i-1}B_{i+1}$. Let $a(p,q,r)$ denote the area of the triangle with vertices p, q, r. Show that

$$\sum_{n=1}^{n-1} \frac{1}{a(A_i, B_i, B_{i-1})} \geq \sum_{n=1}^{n-1} \frac{1}{a(C_i, B_i, B_{i-1})}.$$

Solution. This problem is solved as Theorem 2.1 in [1]. The solution bellow is taken from that reference, and it is based in the following result [Lemma 2.1. in [1]].

Lemma: Let $ABCD$ be a convex quadrilateral and a line through the intersection point O of diagonals AC and BD intersect the sides AD and BC at the points K and L. Then the sum $\dfrac{1}{a(A,O,K)} + \dfrac{1}{a(B,O,L)}$ is minimal if and only if $KL||AB$.





In the situation of the problem, let a line through the point B_i parallel to the line $B_{i-1}B_{i+1}$ intersect the line $A_{i-1}A_i$ at the point A'_i and the line $A_{i+1}A_{i+2}$ at the point A''_{i+1} , and let the lines $A'_{i-1}A''_i$ and $A'_iA''_{i+1}$ meet at C'_i , for $i = 0, 1, \ldots, n-1$ (see Figure 1 where $n = 5$).

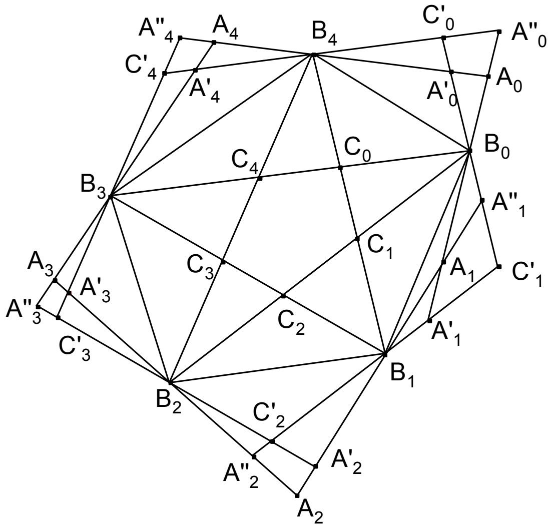


FIGURE 1. The case of the problem for $n = 5$

From the Lemma it follows that

$$\begin{aligned}
\cancel{\frac{1}{a(B_{n-1},A'_0,B_0)}}+\cancel{\frac{1}{a(B_0,A''_1,B_1)}} &\leq \frac{1}{a(B_{n-1},A_0,B_0)}+\frac{1}{a(B_0,A_1,B_1)},\\
\frac{1}{a(B_0,C'_1,B_1)}+\cancel{\frac{1}{a(B_1,A''_2,B_2)}} &\leq \cancel{\frac{1}{a(B_0,A''_1,B_1)}}+\frac{1}{a(B_1,A_2,B_2)},\\
\frac{1}{a(B_1,C'_2,B_2)}+\frac{1}{a(B_2,A''_3,B_3)} &\leq \cancel{\frac{1}{a(B_1,A''_2,B_2)}}+\frac{1}{a(B_2,A_3,B_3)},\\
&\cdots\\
\frac{1}{a(B_{n-3},C'_{n-2},B_{n-2})}+\cancel{\frac{1}{a(B_{n-2},A''_{n-1},B_{n-1})}} &\leq \cancel{\frac{1}{a(B_{n-3},A''_{n-2},B_{n-2})}}+\frac{1}{a(B_{n-2},A_{n-1},B_{n-1})},\\
\frac{1}{a(B_{n-2},C'_{n-1},B_{n-1})}+\frac{1}{a(B_{n-1},C'_0,B_0)} &\leq \cancel{\frac{1}{a(B_{n-2},A''_{n-1},B_{n-1})}}+\cancel{\frac{1}{a(B_{n-1},A'_0,B_0)}}.
\end{aligned}$$

By summing these n inequalities, canceling common terms from both sides and noting that $a(B_iC'_iB_{i-1}) = a(B_iC_iB_{i-1})$ (because the quadrilateral with vertices $B_i, C'_i, B_{i-1}, C_i]$ is a parallelogram, for $i = 0, 1, \ldots, n-1$), the required inequality is obtained. $\square$

REFERENCES

[1] Aliyev Y. N., *New inequalities on triangle areas*, Journal of Qafqaz University, 25 (2009) 129–135.

E-mail address: `aplaza@dmat.ulpgc.es`