

Solution to AMM Problem # 11447

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Problem# 11447 Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania Let a be a positive number, and let g be a continuous, positive, increasing function on $[0, 1]$. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 \left\{ \frac{n}{x} \right\}^a g(x) dx = \frac{1}{a+1} \int_0^1 g(x) dx$$

where $a > 0$ and $\{x\}$ denotes the fractional part of x .

Solution. By substituting $\frac{n}{x} = y$, we obtain

$$\int_0^1 \left\{ \frac{n}{x} \right\}^a g(x) dx = \int_n^\infty \{y\}^a g\left(\frac{n}{y}\right) \frac{n}{y^2} dy = I_n$$

Note that, if $k < y < k + 1$, then $\{y\} = y - k$. Therefore,

$$I_n = \sum_{k=n}^{\infty} \int_k^{k+1} (y - k)^a g\left(\frac{n}{y}\right) \frac{n}{y^2} dy$$

Since g is a continuous, positive, increasing function on $[0, 1]$, then

$$\sum_{k=n}^{\infty} n g\left(\frac{n}{k+1}\right) \int_k^{k+1} \frac{(y - k)^a}{y^2} dy \leq I_n \leq \sum_{k=n}^{\infty} n g\left(\frac{n}{k}\right) \int_k^{k+1} \frac{(y - k)^a}{y^2} dy$$

Let us call $I_k = \int_k^{k+1} (y - k)^a \frac{n}{y^2} dy$. Then, by integrating by parts, we obtain:

$$I_k = \frac{(y - k)^{a+1}}{(a+1)y^2} \Big|_k^{k+1} + \frac{2}{a+1} \int_k^{k+1} \frac{(y - k)^{a+1}}{y^3} dy = \frac{1}{a+1} \left(\frac{1}{(k+1)^2} + 2 \int_k^{k+1} \frac{(y - k)^{a+1}}{y^3} dy \right)$$

Note that $\int_k^{k+1} \frac{(y - k)^{a+1}}{y^3} dy < \int_k^{k+1} \frac{1}{y^3} dy = \frac{-1}{2y^2} \Big|_k^{k+1} = \frac{1}{2k^2} - \frac{1}{2(k+1)^2}$, and then $\frac{1}{(a+1)(k+1)^2} < I_k < \frac{1}{(a+1)k^2}$. Therefore,

$$\frac{1}{a+1} \sum_{k=n}^{\infty} n g\left(\frac{n}{k+1}\right) \frac{1}{(k+1)^2} \leq I_n \leq \frac{1}{a+1} \sum_{k=n}^{\infty} n g\left(\frac{n}{k}\right) \frac{1}{k^2}$$

Since $\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} n g\left(\frac{n}{k+1}\right) \frac{1}{(k+1)^2} = \int_0^1 g(x) dx$, and also $\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} n g\left(\frac{n}{k}\right) \frac{1}{k^2} = \int_0^1 g(x) dx$, the result follows. \square

Note that, by substituting $\frac{n}{x} = y$, we obtain

$$\int_0^1 g(x) dx = \int_n^\infty g\left(\frac{n}{y}\right) \frac{n}{y^2} dy = \sum_{k=n}^{\infty} \int_k^{k+1} g\left(\frac{n}{y}\right) \frac{n}{y^2} dy$$

Since g is a continuous, positive, increasing function on $[0, 1]$, then

$$\sum_{k=n}^{\infty} g\left(\frac{n}{k+1}\right) \frac{n}{(k+1)^2} < \int_0^1 g(x) dx < \sum_{k=n}^{\infty} g\left(\frac{n}{k}\right) \frac{n}{k^2}$$

where, since $\sum_{k=n}^{\infty} g\left(\frac{n}{k}\right) \frac{n}{k^2} - \sum_{k=n}^{\infty} g\left(\frac{n}{k+1}\right) \frac{n}{(k+1)^2} = \frac{g(1)}{n}$, then, taking limits, we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} g\left(\frac{n}{k+1}\right) \frac{n}{(k+1)^2} = \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} g\left(\frac{n}{k}\right) \frac{n}{k^2} = \int_0^1 g(x) dx$$

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