

Solution to AMM Problem 11456

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11456. *Proposed by Raymond Mortini, Université Paul Verlaine, Metz, France. Find*

$$\lim_{n \rightarrow \infty} n \prod_{m=1}^n \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right)$$

SOLUTION. We shall show that the result is $\frac{\cosh(\pi)}{\pi}$

Since $1 - \frac{1}{m} + \frac{5}{4m^2} = \frac{4m^2 - 4m + 5}{4m^2} = \frac{4(m-1/2+i)(m-1/2-i)}{4m^2}$, the proposed limit may be written as

$$L = \lim_{n \rightarrow \infty} n \prod_{m=1}^n \left(1 + \frac{i - \frac{1}{2}}{m}\right) \left(1 + \frac{-i - \frac{1}{2}}{m}\right)$$

Now we will use the product expression of the gamma function:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{e^{-\gamma z}}{z} \prod_{m=1}^n \frac{e^{z/m}}{1 + \frac{z}{m}}$$

to obtain:

$$L = \lim_{n \rightarrow \infty} n \frac{-e^\gamma}{2 \Gamma\left(i - \frac{1}{2}\right) \Gamma\left(-i - \frac{1}{2}\right)} \prod_{m=1}^n e^{\frac{-1}{m}}$$

From the property $\Gamma(x) = \frac{-\pi}{x \Gamma(-x) \sin(\pi x)}$ we get that

$$\Gamma\left(i - \frac{1}{2}\right) \Gamma\left(-i - \frac{1}{2}\right) = \frac{\pi^2}{2 \sin\left(\pi\left(i - \frac{1}{2}\right)\right) \sin\left(\pi\left(i + \frac{1}{2}\right)\right) \Gamma\left(1 - \left(i + \frac{1}{2}\right)\right) \Gamma\left(i + \frac{1}{2}\right)}$$

On the other hand, the use Euler's reflection formula, $\Gamma(x) = \frac{\pi}{\Gamma(1-x) \sin(\pi x)}$ with $x = i + \frac{1}{2}$, leads us to:

$$L = \lim_{n \rightarrow \infty} n \frac{-\sin\left(\pi\left(i - \frac{1}{2}\right)\right) e^\gamma}{\pi} \prod_{m=1}^n e^{\frac{-1}{m}} = \lim_{n \rightarrow \infty} n \frac{\cosh(\pi)}{\pi} e^\gamma \prod_{m=1}^n e^{\frac{-1}{m}}$$

Using again the product expression of the gamma function, we can deduce that:

$$\lim_{n \rightarrow \infty} \prod_{m=1}^n e^{-\frac{z}{m}} = \lim_{n \rightarrow \infty} \frac{e^{-\gamma z}}{z\Gamma(z)} \prod_{m=1}^n \frac{1}{1 + \frac{z}{m}} = \lim_{n \rightarrow \infty} \frac{e^{-\gamma z}}{z\Gamma(z)} \prod_{m=1}^n \frac{m}{z+m}$$

Therefore, making $z = 1$ we see that

$$\lim_{n \rightarrow \infty} n \prod_{m=1}^n \frac{m}{m+1} = e^{-\gamma}$$

and hence

$$L = \lim_{n \rightarrow \infty} n \prod_{m=1}^n \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right) = \frac{\cosh(\pi)}{\pi}$$