Problem 11306 (Proposed by Alexandru Rosoiu, University of Bucharest, Bucharest, Romania)

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Let a, b, and c be the lengths of the sides of a nondegenerate triangle, let $p = (a+b+c)/2$, and let r and R be the inradius and circumradius of the triangle, respectively. Show that

$$\frac{a}{2}\cdot\frac{4r-R}{R} \leq \sqrt{(p-b)(p-c)} \leq \frac{a}{2}$$

and determine the cases of equality.

Solution:

Second inequality is the arithmetic and geometric mean inequality, since $\frac{p-b+p-c}{2} = \frac{a}{2}$. In addition, for this inequality, the only case of equality is $p-b = p-c$, that is $b = c$, or an isosceles triangle.

For the first inequality, note that for the case of an equilateral triangle

$$\frac{a}{2}\cdot\frac{4r-R}{R} = \frac{a}{2}$$

since, in this case, $r = \frac{a}{3\sqrt{3}}$, and $R = \frac{2a}{3\sqrt{3}}$

In general, the circumradius of a triangle is connected to the inradius and the semiperimeter by $R = \frac{abc}{4rp}$. Then

$$\frac{4r-R}{R} = \frac{4r - \frac{abc}{4rp}}{\frac{abc}{4rp}} = \frac{16pr^2}{abc} - 1$$

But if Δ denotes the area of the triangle, then $\Delta = pr$, and also, by Heron's formula $\Delta^2 = p(p-a)(p-b)(p-c)$, from where

$$\frac{4r-R}{R} = \frac{16(p-a)(p-b)(p-c)}{abc} - 1$$

So, the first inequality becomes

$$\frac{a}{2} \cdot \left(\frac{16(p-a)(p-b)(p-c)}{abc} - 1 \right) \leq \sqrt{(p-b)(p-c)}$$

$$\left(\frac{16(p-a)(p-b)(p-c)}{2bc} \right) \leq \sqrt{(p-b)(p-c)} + \frac{a}{2}$$

$$\left(\frac{16(p-a)(p-b)(p-c)}{2bc} \right) \leq \sqrt{(p-b)(p-c)} + \frac{p-b+p-c}{2}$$

Let us rename $p-a = A$, $p-b = B$ and $p-c = C$, then we have to prove

$$\left(\frac{16ABC}{2(A+C)(A+B)} \right) \leq \sqrt{BC} + \frac{B+C}{2}$$

$$16ABC \leq 2(A+C)(A+B) \left(\sqrt{BC} + \frac{B+C}{2} \right)$$

$$16ABC \leq (A+C)(A+B) \left(2\sqrt{BC} + B + C \right)$$

$$16ABC \leq (A+C)(A+B) \left(\sqrt{B} + \sqrt{C} \right)^2 \tag{1}$$

Now, by the arithmetic and geometric mean inequality

$$\sqrt{AB} \leq \frac{A+B}{2};\ \sqrt{AC} \leq \frac{A+C}{2};\ \sqrt{BC} \leq \left(\frac{\sqrt{B}+\sqrt{C}}{2} \right)^2,$$

That is,

$$2\sqrt{AB} \leq A+B;\ 2\sqrt{AC} \leq A+C;\ 4\sqrt{BC} \leq \left(\sqrt{B}+\sqrt{C} \right)^2.$$

So the inequality (1) is proved. Note that the equality holds only if $A = B = C$, that is in the case of an equilateral triangle.