Problem 11299 (Proposed by Pablo Fernández Refolio, Universidad Autónoma de Madrid, Madrid, Spain)

—— Solution by Ángel Plaza
Universidad de Las Palmas de Gran Canaria,
35017-Las Palmas G.C., Spain
aplaza@dmat.ulpgc.es

Show that

$$\prod_{n=2}^{\infty}\left(\frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1}\right)=\frac{e\sqrt{e}}{2\pi}$$

Solution:

Fix an integer $n \geq 2$ and define for integer $i \geq 2$,

$$s(i)=\ln\left(\frac{1}{e}\left(\frac{i^2}{i^2-1}\right)^{i^2-1}\right)=(i^2-1)\ln\left(\frac{i^2}{i^2-1}\right)-1$$

If $S(n)=\sum_{i=2}^{n+1} s(i)$, then $\lim_{n\to\infty} S(n)=\ln\prod_{n=2}^{\infty}\left(\frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1}\right)$, and so the proposed product is equal to e^S, where $S=\lim_{n\to\infty} S(n)$.

$$s(i)=2(i^2-1)\ln(i)-(i^2-1)\ln(i-1)-(i^2-1)\ln(i+1)-1,$$
$$S(n)=\sum_{i=2}^{n+1} s(i)=-n+\sum_{i=2}^{n+1} c(i)\ln(i),$$

with $c(i)$ polynomial functions in i.

We claim

$$c(i)=\left\{\begin{array}{ll} -2 & \text{if } i=2,\\ -2 & \text{for } 3\leq i\leq n,\\ -(n^2-1)+2((n+1)^2-1) & \text{if } i=n+1,\\ -((n+1)^2+1) & \text{if } i=n+2,\\ 0 & \text{for } i>n+2. \end{array}\right\} \tag{1}$$

The proof of (1) is straightforward. For example for $3 \leq i \leq n$ the contribution to $c(i)$ from $s(i)$, $s(i+1)$, and $s(i-1)$ respectively, is $2(i^2-1)$, $-((i+1)^2-1)$, and $-((i-1)^2-1)$ which sums to -2 as required. Proofs of the other cases of (1) are treated similarly.

It follows from (1) that

$$S(n) = \ln\left(\frac{1}{e^n}\frac{1}{n!^2}\left(\frac{n+1}{n+2}\right)^{(n+1)^2-1}(n+1)^{2n+1}\right) \tag{2}$$

To evaluate (2) as $n \to \infty$ we use the following formulae:

$$\left\{\begin{array}{ll} n!^2 \sim (2\pi n)(\frac{n}{e})^{2n} & \text{Stirling's formula,} \\ \lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^{2n+1} = e^2 & \\ \left(\frac{n+1}{n+2}\right)^{(n+1)^2-1} \sim e^{-n-0.5} & \end{array}\right\} \tag{3}$$

The last two equations follow by taking logarithms and using Taylor's formula. For example

$$\lim_{n\to\infty} \ln\left(e^n\left(\frac{n+1}{n+2}\right)^{(n+1)^2-1}\right) = \lim_{n\to\infty}\left(n - \frac{(n+1)^2-1}{n+2} - \frac{1}{2}\frac{(n+1)^2-1}{(n+2)^2} + o(1)\right) = -\frac{1}{2}$$

proving the last formula in (3).

Substituting the limits of (3) into (2) and performing some straightforward cancellations shows

$$S = \lim_{n\to\infty} S(n) = \ln\frac{e\sqrt{e}}{2\pi}$$

from where the proposed product is obtained.