Solution to Problem # B-1055

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Problem# B-1055 *Proposed by G. C. Greubel, Newport News, VA*
Find all integer solutions to the equation

$$x^2 + 6xy + 4y^2 = 4$$

Solution. The equation may be written as $(x + 3y)^2 - 5y^2 = 4$; or, by doing $x = 2(u - 3v)$, and $y = 2v$, as $u^2 - 5v^2 = 1$, which is a Pell's equation. Since 5 is not a perfect square, last equation has infinitely many solutions.

The smallest non-trivial solution of the equation $u^2 - 5v^2 = 1$, where $u > 0$ and $v > 0$, for which x, y are integers, is $(u, v) = (\frac{3}{2}, \frac{1}{2})$. Thus, for every solution (u, v), there is an integer n such that $u + v\sqrt{5} = \pm(\frac{3}{2} + \frac{1}{2}\sqrt{5})^n$. Considering $\phi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, and $\varphi = 1 - \phi = -1/\phi = \frac{1}{2} - \frac{1}{2}\sqrt{5}$, then $\phi^2 = \phi + 1$, and $\varphi^2 = \varphi + 1$, and therefore

$$\begin{aligned} u &= \pm\frac{(1+\phi)^n + (1+\varphi)^n}{2} = \pm\frac{\phi^{2n} + \varphi^{2n}}{2} = \pm\frac{L_{2n}}{2} = \pm\frac{F_{2n+1} + F_{2n-1}}{2} \\ v &= \pm\frac{(1+\phi)^n - (1+\varphi)^n}{2\sqrt{5}} = \pm\frac{\phi^{2n} - \varphi^{2n}}{2\sqrt{5}} = \pm\frac{F_{2n}}{2} \end{aligned}$$

For example, for $u = \dfrac{F_{2n+1} + F_{2n-1}}{2}$ and $v = \dfrac{F_{2n}}{2}$ it is obtained:

$$\begin{aligned} x &= 2u - 6v = F_{2n+1} + F_{2n-1} - 3F_{2n} = -2F_{2n-2} \\ y &= 2v = F_{2n} \end{aligned}$$

Taking into account also the other possibilities for u and v, all integer solutions to the equation $x^2 + 6xy + 4y^2 = 4$ are:

$$\begin{array}{ll} (x, y) = \left(2F_{2n}, -F_{2(n-1)}\right), & (x, y) = \left(2F_{2n}, -F_{2(n+1)}\right) \\ (x, y) = \left(-2F_{2n}, F_{2(n-1)}\right), & (x, y) = \left(-2F_{2n}, F_{2(n+1)}\right) \end{array}$$

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