Solution to Problem # B-1061

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Problem# B-1061 *Proposed by H.-J. Seiffert, Berlin, Germany*
Show that, for all positive integers n,

$$\sum_{k=1}^{n}(-1)^{\left\lfloor \frac{k}{2}\right\rfloor}\frac{F_k}{F_{k+1}}\left(\prod_{j=k}^{n}F_j\right)^2 \quad = \quad (-1)^{\left\lfloor \frac{n-1}{2}\right\rfloor}\frac{F_n}{F_{n+1}}$$

Solution. By induction.
For $n = 1$ the equality becomes trivial. Let us suppose the equality is true for integer n. Then for $n + 1$ we have to prove that

$$\sum_{k=1}^{n+1}(-1)^{\left\lfloor \frac{k}{2}\right\rfloor}\frac{F_k}{F_{k+1}}\left(\prod_{j=k}^{n+1}F_j\right)^2 \quad = \quad (-1)^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{F_{n+1}}{F_{n+2}}$$

We denote by $(LHS)_{n+1}$ the Left-Hand Side (LHS) of the previous equality, that is for the case $n + 1$, and respectively by $(LHS)_n$ for n. Then we have:

$$(LHS)_{n+1} \quad = \quad (F_{n+1})^2\left[(LHS)_n + (-1)^{\left\lfloor \frac{n+1}{2}\right\rfloor}\frac{F_{n+1}}{F_{n+2}}\right]$$

Where, by the induction hypothesis:

$$(LHS)_{n+1} \quad = \quad (F_{n+1})^2\left[(-1)^{\left\lfloor \frac{n-1}{2}\right\rfloor}\frac{F_n}{F_{n+1}} + (-1)^{\left\lfloor \frac{n+1}{2}\right\rfloor}\frac{F_{n+1}}{F_{n+2}}\right]$$

And by the Simson formula,

$$(LHS)_{n+1} \quad = \quad (F_{n+1})^2\,\frac{(-1)^{\left\lfloor \frac{n-1}{2}\right\rfloor}(-1)^{n+1}}{F_{n+1}F_{n+2}} = (-1)^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{F_{n+1}}{F_{n+2}}$$

Which proves the equality for the case $n + 1$. $\square$

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