

Solution to Problem # B-1062

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B-1062 Proposed by M. N. Deshpande, Nagpur, India

Let $g(n) = F_n^2 + F_{n+1}^2 + F_{n+2}^2$ for $n \geq 0$. For every $n \geq 2$, show that $[4g(n+2) - 7g(n+1) - 9g(n)]/4$ is a product of two consecutive Fibonacci numbers.

Solution. We shall prove that

$$(1) \quad [4g(n+2) - 7g(n+1) - 9g(n)]/4 = F_{n-2}F_{n-1}$$

Since $F(n) = (\alpha^n - \beta^n)/\sqrt{5}$, where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F(n)^2 = (\alpha^{2n} + \beta^{2n} - 2(-1)^n)/5$ and therefore

$$\begin{aligned} g(n) &= \frac{\alpha^{2n} + \alpha^{2n+2} + \alpha^{2n+4} + \beta^{2n} + \beta^{2n+2} + \beta^{2n+4} - 2(-1)^n - 2(-1)^{n+1} - 2(-1)^{n+2}}{5} \\ &= \frac{4\alpha^{2n+2} + 4\beta^{2n+2} - 2(-1)^n}{5} \end{aligned}$$

where we have used that $\alpha^2 = \alpha + 1$ and therefore $\alpha^4 + \alpha^2 + 1 = (\alpha + 1)^2 + \alpha^2 + 1 = 2(\alpha^2 + \alpha + 1) = 4\alpha^2$, and also $\beta^4 + \beta^2 + 1 = 4\beta^2$.

Then the left-hand side of Equation (1), LHS , verifies

$$\begin{aligned} 5 \cdot LHS &= 4\alpha^{2n+6} + 4\beta^{2n+6} - 2(-1)^n \\ &\quad - 7\alpha^{2n+4} - 7\beta^{2n+4} + \frac{7}{2}(-1)^{n+1} \\ &\quad - 9\alpha^{2n+2} - 9\beta^{2n+2} + \frac{9}{2}(-1)^{n+2} \\ &= 4L_{2n+6} - 7L_{2n+4} - 9L_{2n+2} - (-1)^n \end{aligned}$$

Applying the recurrence formula $L_{r+1} = L_r + L_{r-1}$ several times it is obtained

$$5 \cdot LHS = L_{2n-3} - (-1)^n$$

And since $5F_m F_k = L_{m+k} - (-1)^k L_{m-k}$, for $m = n - 1$ and $k = n - 2$, we get $5 \cdot LHS = 5F_{n-2}F_{n-1}$. \square