Solution to Problem # B-1063

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B-1063 Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain
Let n be a positive integer. Prove that

$$1+8\sum_{k=1}^{n}\frac{F_{2k}^2}{F_k^2+L_k^2}<\frac{4}{3}\left(F_nF_{n+1}+1\right)\left(L_nL_{n+2}-1\right).$$

Solution. For $n=1$ the inequality reads

$$\begin{aligned}1+8\frac{F_2^2}{F_1^2+L_1^2} &< \frac{4}{3}\left(F_1F_2+1\right)\left(L_1L_3-1\right)\\ 1+8\frac{1}{1+1} &< \frac{4}{3}\cdot 6\\ 1+4 &< 8\end{aligned}$$

In the following we use that $F_{2k}=F_kL_k$:

$$\begin{aligned}1+8\sum_{k=1}^{n}\frac{F_{2k}^2}{F_k^2+L_k^2} &= 1+8\sum_{k=1}^{n}\frac{F_k^2L_k^2}{F_k^2+L_k^2}<1+8\sum_{k=1}^{n}\frac{F_k^2L_k^2}{2F_k^2}\\ &< 1+4\sum_{k=1}^{n}L_k^2=1+4\left(L_nL_{n+1}-2\right)\\ &= 1+4\left(L_nL_{n+2}-L_n^2-2\right)<4\left(L_nL_{n+2}-1\right).\end{aligned}$$

Now tanking into account that for $n>1$, $4\leq\frac{4}{3}\left(F_nF_{n+1}+1\right)$, the problem is done. $\square$

It should be noted that the proposed inequality may be improved, for example, in the following way, and with a similar proof:

For any integer $n\geq 2$ it holds

$$1+8\sum_{k=1}^{n}\frac{F_{2k}^2}{F_k^2+L_k^2}<L_nL_{n+2}.$$