Solution to Problem # B-1064

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B-1064 Proposed by N. Gauthier, Kingston, ON, Canada

For $a \neq 0$, let $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = af_{n+1} + f_n$ for $n \geq 0$. If n is a positive integer, find a closed-form expression for

$$\sum_{k=0}^{n-1} f_k^3$$

Solution. We shall prove the following closed-form expression:

$$\sum_{k=0}^{n-1} f_k^3 = \frac{f_{3n} + f_{3n-3} + 1}{f_4 + f_2} - \frac{f_n^3 + f_{n-1}^3}{a} \tag{1}$$

Lemma: For $n \geq 0$ it holds

$$f_n^3 + \frac{f_{n+1}^3 - f_{n-1}^3}{a} = f_{3n}. \tag{2}$$

The closed-form expression follows easily from the Lemma. Note that summing up in the Lemma we obtain:

$$\sum_{k=0}^{n-1} f_k^3 + \frac{1}{a}\sum_{k=0}^{n-1}\left(f_{k+1}^3 - f_{k-1}^3\right) \;=\; \sum_{k=0}^{n-1} f_{3k}$$

Using that the second sum is telescopic and the expression for the sum of numbers f_{3k} with indexes in an arithmetic sequence [2] it is readily obtained:

$$\sum_{k=0}^{n-1} f_k^3 + \frac{1}{a}\left(f_n^3 + f_{n-1}^3 - f_0^3 - f_{-1}^3\right) \;=\; \frac{f_{3n} + f_{3n-3} - f_3}{f_4 + f_2}$$

Identity (1) follows since $f_0 = 0$, $f_{-1} = (-1)^2 f_1 = f_1 = 1$, and $\dfrac{1}{a} - \dfrac{f_3}{f_4 + f_2} = \dfrac{1}{f_4 + f_2}$.
$\square$

Proof of the lemma: We follow here a combinatorial argument taken from Benjamin and Quinn [1, Identity 232, pp. 126-127] and adapt it for the case of distinguished squares.

It is well known that the numbers of this problem, f_n, count the number of tilings of an $(n-1)$-board with a-distinguished (or colored) squares and black dominoes [1]. For convenience, we will use the notation $F_n = f_{n+1}$. For a-distinguished squares we understand that each square may be labeled (or colored) in k different ways. Therefore, is this notation the identity to be proved becomes

$$F_{n-1}^3 + \frac{F_n^3 - F_{n-2}^3}{a} = F_{3n-1} \tag{3}$$

We use the concepts of *breakable* tiling and *unbreakable* tiling [1]. It is said that a tiling of an n-board is *breakable* at cell p, if the tiling can be decomposed into two tilings, one covering cells 1 through p and the other covering cells $p+1$ through n. On the other hand, a tiling is said to be *unbreakable* at cell p if a domino occupies cells p and $p+1$. See Fig. 1.

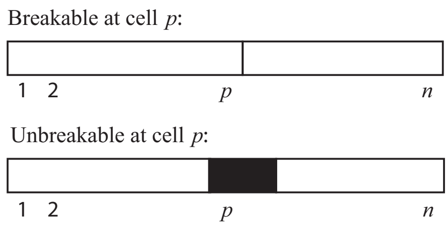

FIGURE 1. An (n)-board is either breakable or unbreakable at cell p

For proving identity (3) we shall define two sets and a correspondence between them:

Set 1: The set of ordered triples (A, B, C), where A, B, and C are $(n-1)$-tilings or A, B, and C are n-tilings with at least one ending with a square. Discarding the triples of n-tilings that all end in a domino shows that this set has size $F_{n-1}^3 + F_n^3 - F_{n-2}^3$.

Set 2: The set of $(3n-1)$-tilings. There are F_{3n-1} such tilings.

Correspondence: We basically to distinguish two different cases. If (A, B, C) is a triple of $(n-1)$-tilings, we generate a $(3n-1)$-tiling that is unbreakable at cells $n-1$ and unbreakable at cell $(2n-1)$ by appending B the C to A with an extra domino inserted between B and C. See Fig. 2.

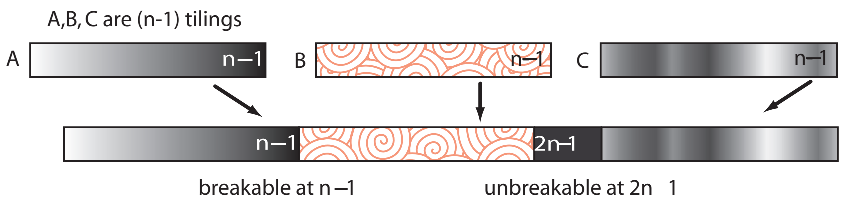

FIGURE 2. Correspondence between ordered $(n-1)$ tilings (A, B, C) and $(3n-1)$-tilings.

In other case, if (A, B, C) is a triple of n-tilings with at least one ending with a square. We proceed as follows. If A ends in a square, then we append B then C to A after removing the last square of A. This creates, for each a of such triples, one $(3n-1)$-tiling that is breakable at cell $n-1$ and $2n-1$. If A ends in a domino and B ends in a square, then we append B, then C to A after removing the last square of B. This creates, for each a of such triples, one $(3n-1)$-tiling that is unbreakable at cell $n-1$ and breakable at cell $2n-1$. If A and B end with a domino and C ends with a square, then we do the same as before, bu tnow removing the last square of C. This gives us one, for each a of such triples, one $(3n-1)$-tiling that is breakable at cell $n-1$ and unbreakable at cell $2n-1$. The correspondence is illustrated in Fig. 3 taken also from [1]. Note that in the first case the correspondence is one to one while in the second case, which in fact consists in three sub-cases, the correspondence is a to one because an a-colored square is removed. This explains the division by a in Equation (3). □

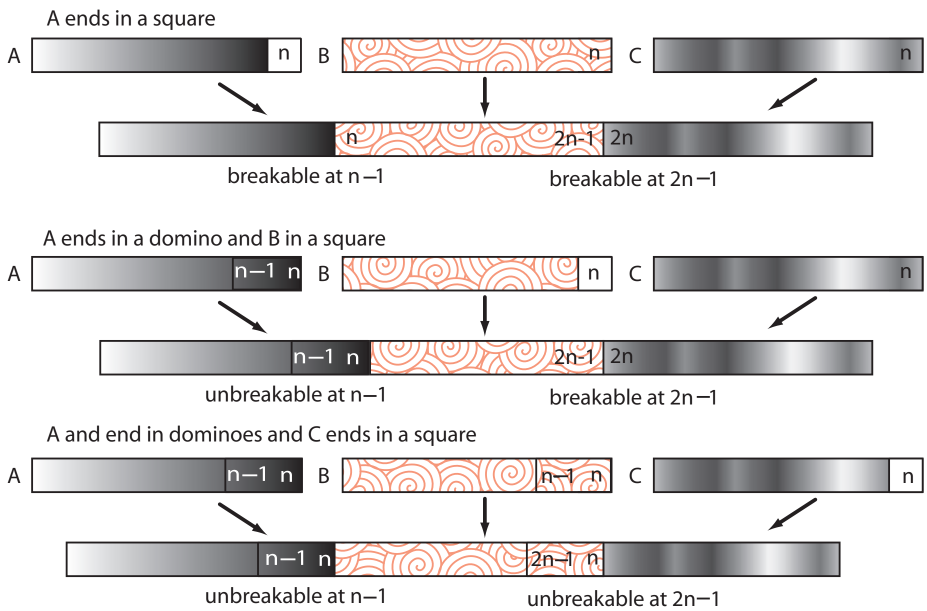

FIGURE 3. Correspondence between ordered n tilings (A, B, C) with at least one ending with a square and $(3n-1)$-tilings.

REFERENCES

[1] A. T. Benjamin, J. J. Quinn, *Proofs That Really Count, The Art of Combinatorial Proof*, MAA, 2003.

[2] S. Falcón, A. Plaza, *On k-Fibonacci numbers of arithmetic indexes*, Applied Mathematics and Computation, 208 (1) (2009) 180-185.