# Solution to Problem # B-1065

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**B-1065 Proposed by Br. J. Mahon, Australia**

The Pell numbers $P_n$ satisfy $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$. Prove that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1} P_{6r+3}}{P_{3r}^2 P_{3r+3}^2} = \frac{1}{125}.$$

**Solution.** We consider the sequence of general term $Q_m = P_{3m}$. The new sequence is again a generalized Fibonacci sequence (see for example [1]) verifying the recurrence relation $Q_{n+2} = 14Q_{n+1} + Q_n$ with initial values $Q_0 = 0$ and $Q_1 = 5$. The Binet's formula for sequence $\{Q_n\}$ is $Q_n = 5\dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, being $\alpha$ and $\beta$ the roots of the characteristic polynomial. By the Binet's formula it is deduced the convolution product: $Q_{n+m} = \frac{Q_{n+1}Q_m + Q_n Q_{m-1}}{5}$. Note that $P_{6r+3} = Q_{2r+1}$, and using the convolution product for $n = r$ and $m = r + 1$ it is readily obtained

$$\begin{aligned} \frac{P_{6r+3}}{P_{3r}^2 P_{3r+3}^2} &= \frac{Q_{2r+1}}{Q_r^2 Q_{r+1}^2} = \frac{Q_r^2 + Q_{r+1}^2}{5Q_r^2 Q_{r+1}^2} \\ &= \frac{1}{5Q_r^2} + \frac{1}{5Q_{r+1}^2}. \end{aligned}$$

Therefore,

$$\sum_{r=1}^{\infty} (-1)^{r-1} \left( \frac{1}{5Q_r^2} + \frac{1}{5Q_{r+1}^2} \right) = \frac{1}{5Q_1^2} = \frac{1}{125}$$

$\square$

## References

[1] S. Falcón, A. Plaza, *On $k$-Fibonacci numbers of arithmetic indexes*, Applied Mathematics and Computation, 208 (1) (2009) 180-185.