Solution to Problem # H-668

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Problem# H-668 *Proposed by A. Cusumano, Great Neck, NY*
For each $k \geq 2$, let $\left(F_n^{(k)}\right)_{n\geq 1}$ be the kth order linear recurrence given by

$$F_{n+k}^{(k)} = \sum_{i=0}^{k-1} F_{n+i}^{(k)}, \text{ for all } n \geq 1,$$

with $F_n^{(k)} = 1$ for $n = 1, \ldots, k$. Prove the following:

(a) $R_k = \lim_{n\to\infty} F_{n+1}^{(k)}/F_n^{(k)}$ exists for all $k \geq 1$.
(b) $\lim_{k\to\infty} R_k = 2$.
(c) $\lim_{k\to\infty} (R_{k+1} - R_k) / (R_{k+2} - R_{k+1}) = 2$.

Solution. We use the following representation of the nth k-generalized Fibonacci number [1, Theorem 1]:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}$$

for $\alpha_1, \ldots, \alpha_k$ the roots of $x^k - x^{k-1} - \cdots - 1 = 0$.

Moreover, equation $x^k - x^{k-1} - \cdots - 1 = 0$ has just one root α such that $|\alpha| > 1$, and the other roots are strictly inside the unit circle [1]. Therefore, the contribution of the other roots in the formula for $F_n^{(k)}$ will quickly become trivial, and thus $F_n^{(k)} \sim \dfrac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1}$, for n suffciently large.

(a) $R_k = \lim_{n\to\infty} F_{n+1}^{(k)}/F_n^{(k)}$ exists for all $k \geq 1$:

$$R_k = \lim_{n\to\infty} \frac{F_{n+1}^{(k)}}{F_n^{(k)}} = \lim_{n\to\infty} \frac{\frac{\alpha-1}{2+(k+1)(\alpha-2)}\alpha^n}{\frac{\alpha-1}{2+(k+1)(\alpha-2)}\alpha^{n-1}} = \alpha$$

Let us rename α_k the unique root of equation $x^k - x^{k-1} - \cdots - 1 = 0$ such that $|\alpha_k| > 1$. Then, we have obtained that $R_k = \alpha_k$.

(b) $\lim_{k\to\infty} R_k = 2$:

In the same reference [1] it is established that for $k \geq 4$, $2 - \dfrac{1}{3k} < \alpha_k < 2$ holds. Then,

$$\lim_{k\to\infty} 2 - \frac{1}{3k} \leq \lim_{k\to\infty} R_k = \lim_{k\to\infty} \alpha_k \leq 2$$
$$\lim_{k\to\infty} R_k = 2.$$

(c) $L = \lim_{k\to\infty} (R_{k+1} - R_k) / (R_{k+2} - R_{k+1}) = 2$:

This may be written as $L = \lim_{k\to\infty} (\alpha_{k+1} - \alpha_k) / (\alpha_{k+2} - \alpha_{k+1}) = 2$: We use now the following theorem [2, Theorem 3.9]: *For each positive integer k, let $2(1-\epsilon_k)$ be the positive root of the respective characteristic equation. Then*

$$\epsilon_k = \sum_{i\geq 1} \binom{(k+1)i-2}{i-1} \frac{1}{i2^{(k+1)i}}$$

Then the limit becomes

$$\begin{aligned} L &= \lim_{k\to\infty} \frac{\epsilon_{k+1} - \epsilon_k}{\epsilon_{k+2} - \epsilon_{k+1}} \\ &= \lim_{k\to\infty} \frac{\sum_{i\geq 1} \binom{(k+2)i-2}{i-1} \frac{1}{i2^{(k+2)i}} - \sum_{i\geq 1} \binom{(k+1)i-2}{i-1} \frac{1}{i2^{(k+1)i}}}{\sum_{i\geq 1} \binom{(k+3)i-2}{i-1} \frac{1}{i2^{(k+3)i}} - \sum_{i\geq 1} \binom{(k+2)i-2}{i-1} \frac{1}{i2^{(k+2)i}}} \\ &= \lim_{k\to\infty} \frac{\frac{1}{2^{k+2}} - \frac{1}{2^{k+1}}}{\frac{1}{2^{k+3}} - \frac{1}{2^{k+2}}} = 2 \end{aligned}$$

□

REFERENCES

[1] G. P. B. Desden, *A Simplified Binet Formula for k-Generalized Fibonacci Numbers*, arXiv:0905.0304v1.

[2] D. A. Wolfram, *Solving Generalized Fibonacci Recurrences*, The Fibonacci Quarterly, 36(2) (May 1998): 129-145.

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