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Prove that for any positive integer, n,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}. \tag{1}$$

Solution: *(by Ángel Plaza and Sergio Falcón, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)*

Note that the proposed identity may be written as

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}}{(2n+1)\binom{2n}{n}^2} \tag{2}$$

Since:

$$\frac{\binom{n}{k}^2\binom{2n}{n}}{\binom{2n}{2k}} = \frac{\frac{(n!)^2}{(k!)^2((n-k)!)^2} \cdot \frac{(2n)!}{(n!)^2}}{\frac{(2n)!}{(2k)!(2n-2k)!}} = \binom{2k}{k}\binom{2n-2k}{n-k}$$

Equation (2) is equivalent to:

$$\binom{2n}{n} \sum_{k=0}^{n} \frac{\binom{2k}{k}\binom{2n-2k}{n-k}}{2k+1} = \frac{2^{4n}}{2n+1} \tag{3}$$

By using the *falling factorial powers* of real x defined by $x^{\underline{k}} = x(x-1)\dots(x-k+1)$, where $x^{\underline{0}} = 1$ yields [1, Eq.(5.36)]:

$$\binom{m-1/2}{m} = \binom{2m}{m} \Big/ 2^{2n},$$

and therefore:

$$\binom{2m}{m} = \binom{m-1/2}{m} 2^{2m}$$

Hence, Equation (3) is equivalent to:

$$\left(n-\frac{1}{2}\right)^{\underline{n}} \sum_{k=0}^{n} \frac{(k-\frac{1}{2})^{\underline{k}}\left(n-k-\frac{1}{2}\right)^{\underline{n-k}}}{(2k+1)k!(n-k)!} = \frac{n!}{2n+1} \tag{4}$$

Now, since $\left(n+\frac{1}{2}\right)^{\underline{n+1}} = \left(n+\frac{1}{2}\right)^{\underline{n}}\frac{1}{2} = \left(n+\frac{1}{2}\right)^{\underline{n-k}}\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)^{\underline{k}}$, then $\frac{1}{2k+1} = \frac{\left(n+\frac{1}{2}\right)^{\underline{n-k}}\left(k-\frac{1}{2}\right)^{\underline{k}}}{\left(n+\frac{1}{2}\right)^{\underline{n}}}$, so we get

$$\frac{\left(n-\frac{1}{2}\right)^{\underline{n}}}{\left(n+\frac{1}{2}\right)^{\underline{n}}}\cdot\sum_{k=0}^{n}\frac{\left[\left(k-\frac{1}{2}\right)^{\underline{k}}\right]^2\left(n-k-\frac{1}{2}\right)^{\underline{n-k}}\left(n+\frac{1}{2}\right)^{\underline{n-k}}}{k!(n-k)!} = \frac{n!}{2n+1} \tag{5}$$

Note that $\frac{\left(n-\frac{1}{2}\right)^{\underline{n}}}{\left(n+\frac{1}{2}\right)^{\underline{n}}} = \frac{1}{2n+1}$, so we have:

$$\sum_{k=0}^{n}\frac{\left[\left(k-\frac{1}{2}\right)^{\underline{k}}\right]^2\left(n-k-\frac{1}{2}\right)^{\underline{n-k}}\left(n+\frac{1}{2}\right)^{\underline{n-k}}}{k!(n-k)!} = n! \tag{6}$$

Now we use $\left(k-\frac{1}{2}\right)^{\underline{k}} = \left(\frac{-1}{2}\right)^{\underline{k}}(-1)^k$, to obtain:

$$\sum_{k=0}^{n}\binom{n}{k}\left[\left(\frac{-1}{2}\right)^{\underline{k}}(-1)^k\right]^2\left(\frac{-1}{2}\right)^{\underline{n-k}}(-1)^{n-k}\left(n+\frac{1}{2}\right)^{\underline{n-k}} = (n!)^2$$

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}\left[\left(\frac{-1}{2}\right)^{\underline{k}}\right]^2\left(\frac{-1}{2}\right)^{\underline{n-k}}\left(n+\frac{1}{2}\right)^{\underline{n-k}}(-1)^k = (n!)^2$$

And the last equation follows from the Pfaff formula:

$$\sum_{k=0}^{n}\binom{n}{k}(a_1)^{\underline{k}}(a_2)^{\underline{k}}(b_1)^{\underline{n-k}}(b_2)^{\underline{n-k}}(-1)^k = (a_1+b_1)^{\underline{n}}(a_2+b_1)^{\underline{n}}(-1)^n$$

where $a_1+a_2+b_1+b_2 = n-1$.

In our case, $a_1 = a_2 = b_1 = \frac{-1}{2}$, and $b_2 = n+\frac{1}{2}$.

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics. Addison-Wesley Pub. Co. Second Ed. (1998).