SOLUTION TO AMM PROBLEM # 11369

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Problem . # 11369 *Show that for all real t, and all $\alpha \geq 2$,*

$$e^{\alpha t} + e^{-\alpha t} - 2 \leq \left(e^t + e^{-t}\right)^\alpha - 2^\alpha.$$

Solution: It is clear that the equality holds for $t = 0$ and any $\alpha \geq 2$, and also for any real t and $\alpha = 2$. Let us suppose then that $t \neq 0$ and $\alpha > 2$. Since $x = e^t > 0$ in this case, the inequality may be written as

(1)
$$x^\alpha + x^{-\alpha} - 2 < \left(x + \frac{1}{x}\right)^\alpha - 2^\alpha.$$

Also, since $x \cdot x^{-1} = 1$ it can be supposed that $x > 1$.

Note that if $g(x) = x^\alpha + x^{-\alpha}$ and $f(x) = \left(x + \frac{1}{x}\right)^\alpha$, then Eq. (1) may be written as

(2)
$$g(x) - g(1) < f(x) - f(1),$$

or, equivalently,

(3)
$$\frac{g(x) - g(1)}{f(x) - f(1)} < 1.$$

Now, by the Lagrage Theorem, the Left-Hand Side of Eq. (3) is $\dfrac{g'(c)}{f'(c)}$, for some real c such that $1 < c < x$.

Note that $\dfrac{g'(c)}{f'(c)} < 1 \quad \Leftrightarrow g'(c) < f'(c)$. That is, using x instead of c,

(4)
$$\alpha x^{\alpha-1} - \alpha \frac{1}{x^{\alpha+1}} < \alpha \left(x + \frac{1}{x}\right)^{\alpha-1} \left(1 - \frac{1}{x^2}\right)$$

(5)
$$x^{\alpha-1} \left[1 - \frac{1}{x^{2\alpha}}\right] < x^{\alpha-1} \left(1 + \frac{1}{x^2}\right)^{\alpha-1} \left(1 - \frac{1}{x^2}\right)$$

$\dfrac{1}{x^2} = y$ gives $0 < y < 1$ and Eq. (5) reads:

(6)
$$1 - y^\alpha < (1+y)^{\alpha-1}(1-y) = (1+y)^{\alpha-1} - y(1+y)^{\alpha-1}$$

(7)
$$1 - (1+y)^{\alpha-1} < y^\alpha - y(1+y)^{\alpha-1} = y\left[y^{\alpha-1} - (1+y)^{\alpha-1}\right]$$

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$$\frac{(1+y)^{\alpha-1}-1}{(1+y)^{\alpha-1}-y^{\alpha-1}} > y \tag{8}$$

Let us consider function $F(y) = y^{\alpha-1}$. F is strictly convex, since $F''(y) = (\alpha-1)(\alpha-2)y^{\alpha-3} > 0$, for $y > 0$ and $\alpha > 2$. If denote by $\Delta_F(x,y) = \dfrac{F(y)-F(x)}{y-x}$ the divided difference of function F, then Eq. (8) may be understood as:

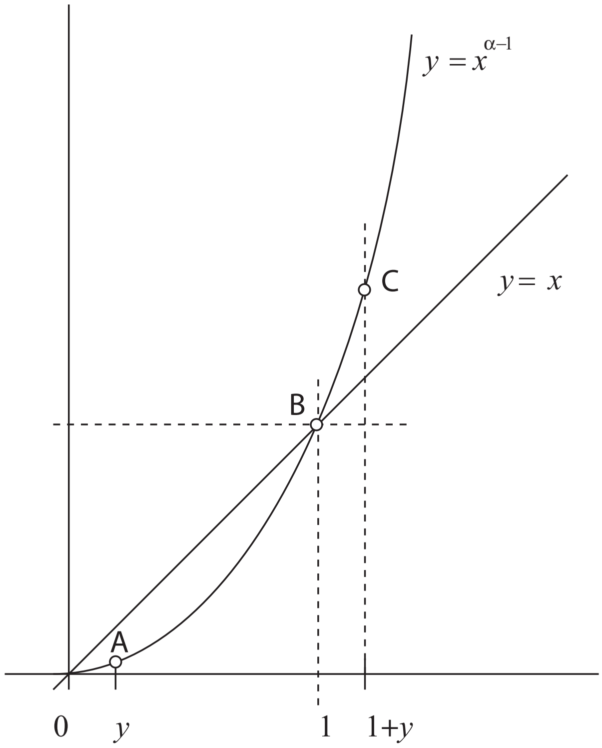
$$y\frac{\Delta_F(1,1+y)}{\Delta_F(y,1+y)} > y \tag{9}$$

which is equivalent to

$$\frac{\Delta_F(1,1+y)}{\Delta_F(y,1+y)} > 1 \ \Leftrightarrow\ \Delta_F(1,1+y) > \Delta_F(y,1+y) \tag{10}$$

Now, we use the following lemma [1]:
LEMMA. *A function $F : (a,b) \to R$ is convex (strictly convex) if and only if its divided difference $\Delta_F(x,y)$ is increasing (strictly increasing) in both variables.*

Inequality (10) may be illustrated by the following figure, considering that $\Delta_F(1,1+y)$ is the slope of the line passing through points B and C, while $\Delta_F(y,1+y)$ is the slope of the line passing through points A and C:



Note, also, that for the case $\alpha = 2$, function $y = x^{\alpha-1}$ into the previous figure is precisely $y = x$ and in this case we have the equality. $\square$

REFERENCES

1. Z. Kadelbur, D. Duckić, M. Lukić, I. Matić, *Inequalities of Karamata, Schur and Muirhead, and some applications*, The Teaching of Mathematics, vol. VIII (1) (2005), 31–45.