Problem # 11381. Proposed by Jesús Guillera, Zaragoza, Spain, and Jonathan Sondow, New York, NY.

Show that if x is a positive real number, then

$$e^x = \prod_{n=1}^{\infty} \left(\prod_{k=1}^{n} (kx+1)^{(-1)^{k+1}\binom{n}{k}} \right)^{1/n} \tag{1}$$

Solution: *(by Ángel Plaza, and Sergio Falcón, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)*

The identity is proved in the paper:

J.Guillera and J. Sondow, Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent, The Ramanujan J., Vol. 16, No. 3 (2008), 247-270.

The proof is based on the following result (Theorem 5.2.), also given in the same reference:

Let us recall the formulas for the *Euler beta function*:

$$B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1}dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

Theorem 5.2. For $j = 1, 2, \ldots,$

$$B(u,j) = \sum_{n=j}^{\infty} \frac{1}{n-j+1} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \ln(u+k). \tag{2}$$

□

Now the proof of Eq. (1) follows as:

The formula (1) holds when $x = 0$. If $x > 0$, then since u, unlike j, is not restricted to integer values in Theorem 5.2, we may take $u = 1/x$. Since $\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k} = 0$, we may then replace

$\ln\left(\frac{1}{2}+k\right)$ with $\ln(kx+1)$ in Eq. (1). Setting $j = 1$ and using $B(1/x, 1) = x$, we exponentiate Eq. (1) and obtain the result. $\square$