

SOLUTION TO AMM PROBLEM # 11383

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Problem# 11383 *Proposed by Michael Nyblom, RMIT University, Melbourne, Australia.*

Show that

$$\sum_{n=1}^{\infty} \cos^{-1} \left(\frac{1 + \sqrt{n^2 + 2n} \sqrt{n^2 + 4n + 3}}{(n+1)(n+2)} \right) = \frac{\pi}{3}$$

Solution. I think that the solution should be $\frac{\pi}{6}$ instead of $\frac{\pi}{3}$.

Let us denote $c_n = \frac{1 + \sqrt{n^2 + 2n} \sqrt{n^2 + 4n + 3}}{(n+1)(n+2)}$. We have to prove that

$$\sum_{n=1}^{\infty} \cos^{-1} c_n = \frac{\pi}{6}.$$

$$c_n = \frac{1}{n+1} \frac{1}{n+2} + \sqrt{1 - \left(\frac{1}{n+1} \right)^2} \sqrt{1 - \left(\frac{1}{n+2} \right)^2}$$

Therefore

$$\begin{aligned} \cos^{-1} c_n &= \cos^{-1} \left(\sqrt{1 - \left(\frac{1}{n+1} \right)^2} \right) - \cos^{-1} \left(\sqrt{1 - \left(\frac{1}{n+2} \right)^2} \right) \\ &= \sin^{-1} \left(\frac{1}{n+1} \right) - \sin^{-1} \left(\frac{1}{n+2} \right) \end{aligned}$$

Hence, after cancelling terms in the telescopic sum it is obtained

$$\sum_{n=1}^N \cos^{-1} \left(\frac{1 + \sqrt{n^2 + 2n} \sqrt{n^2 + 4n + 3}}{(n+1)(n+2)} \right) = \sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} \left(\frac{1}{N+2} \right)$$

Since $\lim_{N \rightarrow \infty} \sin^{-1} \left(\frac{1}{N+2} \right) = 0$, then

$$\sum_{n=1}^{\infty} \cos^{-1} \left(\frac{1 + \sqrt{n^2 + 2n} \sqrt{n^2 + 4n + 3}}{(n+1)(n+2)} \right) = \lim_{N \rightarrow \infty} \sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} \left(\frac{1}{N+2} \right) = \frac{\pi}{6}$$

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