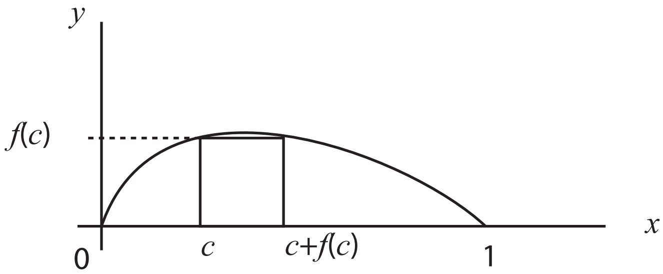
Problem # 11402. Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania.

Let $f : [0,1] \to [0,\infty)$ be a continuos function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval $(0,1)$ on the x -axis and the other two vertices on the graph of f.

Solution: *(by Ángel Plaza, and Sergio Falcón, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)*



Extend f continuously to $[0,\infty)$ by defining $f(x) = 0$ for $x > 1$. Consider the functions $g(x) = x+f(x)$ and $h(x) = f(x)-f(g(x)) = f(x) - f(x + f(x))$ both also defined and continuous on $[0,\infty)$. We seek a point $c \in (0,1)$ with $h(c) = 0$, for then the vertices $(c,0)$, $(c + f(c), 0)$, $(c + f(c), f(c + f(c)))$, and $(c, f(c))$ form the desired square, with all sides equal to $f(c)$. See the figure.

Since f is continuous on $[0,1]$ it takes its (positive) maximum there at some point $M \in (0,1)$. Since M is also a maximum of the extended f we have $f(M) > f(M + f(M))$, so $h(M) > 0$. On the other hand, since $g(0) = 0$, $g(1) = 1$ and g is continuous, the connected set $g([0,1])$ contains (at least) the interval $[0,1]$, so there is an m with $g(m) = M$, that is $m + f(m) = M$, so $m = M - f(m) < M$. Then $f(m) < f(M) = f(g(m))$, so $h(m) < 0$. By the Intermediate Value Theorem, $h(c) = 0$ for some $0 < m < c < M < 1$, and the proof is complete. $\square$