

Problem # 11403. Proposed by Yaming Yu, University of California Irvine, Irvine, CA.

Let n be an integer greater than 1, and let f_n be the polynomial given by

$$f_n(x) = \sum_{i=0}^n \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x+j)$$

Find the degree of f_n .

Solution: (by Ángel Plaza, and Sergio Falcón, University of Las Palmas de Gran Canaria, 35017-Las Palmas G.C., Spain)

Since the Stirling numbers can be defined as coefficients in the expansion of the rising factorial function (Louis Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Boston, 1974.):

$$x(x+1)(x+2)\dots(x+i-1) = \sum_{m=1}^i \left[\begin{matrix} n \\ m \end{matrix} \right] x^m.$$

the polynomial f_n can be written as

$$f_n(x) = \sum_{i=0}^n \binom{n}{i} (-x)^{n-i} \sum_{m=1}^i \left[\begin{matrix} i \\ m \end{matrix} \right] x^m.$$

Therefore, the coefficient of x^k in polynomial f_n will be

$$a_{n,k} = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \left[\begin{matrix} i \\ k-n+i \end{matrix} \right] = \sum_{i=0}^n \binom{n}{i} (-1)^i \left[\begin{matrix} n-i \\ k-i \end{matrix} \right]$$

For integers $n \geq k \geq 0$, $\left[\begin{matrix} n \\ m \end{matrix} \right]$ counts the number of permutations of n elements with exactly m cycles. Equivalently $\left[\begin{matrix} n \\ m \end{matrix} \right]$ counts the number of ways for n distinct people to sit around m identical circular tables, where no tables are allowed to be empty (A.T. Benjamin & J.J. Quinn, *Proofs that really count*, The Mathematical Association of America, 2003). Therefore, $\binom{n}{i} \left[\begin{matrix} n-i \\ k-i \end{matrix} \right]$ counts the number of ways for n people to sit around $k-i$ identical circular tables, once that i people are decided to sit alone. In this sense, $a_{n,k}$ is the number of ways to n people to sit around in such a way that there are k circular tables with more than 1 people. It follows that $a_{n,k} = 0$ if $2k > n$, that is if $k > \lfloor n/2 \rfloor$. In addition, if $k = \lfloor n/2 \rfloor$, then $a_{n,k} \neq 0$ and hence the degree of f_n is $\lfloor n/2 \rfloor$ \square