

### Solution to AMM Problem # 11418

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**Problem# 11418** *Proposed by George Lamb, Tucson, AZ.* Find

$$\int_{-\infty}^{\infty} \frac{t^2 \operatorname{sech}^2 t}{a - \tanh t} dt$$

for complex  $a$  with  $|a| > 1$ .

**Solution.** By using the exponential form of  $\operatorname{sech} t$  and  $\tanh t$ , we get

$$I = \int_{-\infty}^{\infty} \frac{t^2 \operatorname{sech}^2 t}{a - \tanh t} dt = \int_{-\infty}^{\infty} \frac{4e^{2t} t^2}{(1 + e^{2t})(1 + a + (a - 1)e^{2t})} dt$$

By the change of variable  $u = e^{2t}$  we obtain

$$I = \frac{1}{2} \int_0^{\infty} \frac{\ln^2 u}{(1 + u)(1 + a + (a - 1)u)} du$$

Note that if  $F(s) = \frac{1}{2} \int_0^{\infty} \frac{u^s}{(1 + u)(1 + a + (a - 1)u)} du$ , then  $F''(0) = I$ .

$$F(s) = \frac{1}{4} \int_0^{\infty} \frac{u^s}{1 + u} du + \frac{1-a}{4} \int_0^{\infty} \frac{u^s}{1 + a + (a - 1)u} du$$

Now, if we suppose that  $-1 < \operatorname{Re} s < 0$  and  $|a| > 1$  it is obtained

$$\begin{aligned} \frac{1}{4} \int_0^{\infty} \frac{u^s}{1 + u} du &= -\frac{\pi}{4} \csc(\pi s) \\ \frac{1-a}{4} \int_0^{\infty} \frac{u^s}{1 + a + (a - 1)u} du &= -\frac{\pi}{4}(1-a)(a-1)^{-s-1}(a+1)^s \csc(\pi s) \end{aligned}$$

And therefore,  $F(s) = \frac{\pi}{4} \csc(\pi s) \left( \left( \frac{a+1}{a-1} \right)^s - 1 \right)$ . Here, by doing  $\lim_{s \rightarrow 0} F''(s)$  it is obtained

$$I = \frac{1}{12} \ln^3 \left( \frac{a+1}{a-1} \right) + \frac{\pi^2}{12} \ln \left( \frac{a+1}{a-1} \right)$$

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