Solution to AMM Problem # 11423

Ángel Plaza
Department of Mathematics, Universidad de Las Palmas de Gran Canaria
35017–Las Palmas G.C. SPAIN,

Problem# 11423 *Proposed by Gregory Minton, D. E. Shaw Research, LLC, New York, NY.* Show that if n and m are positive integers with $n \geq m$ and $n - m$ even, then $\int_{x=0}^{\infty} x^{-m} \sin^n x \, dx$ is a rational multiple of π.

Solution. $\int_0^\infty f(x)\, dx$ is defined as the limit $\lim\limits_{R\to\infty} \displaystyle\int_0^R f(x)\, dx$, provided that this limit exists. When the function $f(x)$ is even, as in our case, one has $\displaystyle\int_0^R f(x)\, dx = \frac{1}{2}\int_{-R}^R f(x)\, dx$.

In our case, since $\sin x = \Im\left(e^{iz}\right)$, and $\cos nz + i \sin nz = \left(e^{iz}\right)^n$, then $\sin^n z$ may be written as the real part if n is even (or imaginary part if n is odd) of a polynomial on e^{iz} of degreee n with rational coefficients. For our purposes, we consider here an integral of the form $\lim\limits_{R\to\infty}\left(\displaystyle\int_R^R \frac{e^{inz}}{z^m}\, dz\right)$.

Let us consider the complex integral $I_R = \int_{C_R} \frac{e^{inz}}{z^m}\, dz$, where $C_R = [-R, -\epsilon] \cup \delta_\epsilon \cup [\epsilon, R] \cup \delta_R$. That is, C_R is a contour consisting on two line segments along the real axis, between $-R$ and ϵ, and between ϵ and R, and two semicircles centered at the origin: $\delta_\epsilon = \{z = \epsilon e^{it},\ t \in [\pi, 0]\}$, and $\delta_R = \{z = Re^{it},\ t \in [0, \pi]\}$.

By the Residue Theorem, $I_R = 0$ since there is no interior pole to C_R of function $\frac{e^{inz}}{z^m}$. On the other hand, for $\epsilon \to 0$, and $R \to \infty$ we have $\displaystyle\int_{[-R,-\epsilon]\cup[\epsilon,R]} \frac{e^{inz}}{z^m}\, dz \longrightarrow \lim\limits_{R\to\infty}\left(\int_R^R \frac{e^{inx}}{x^m}\, dx\right)$, $\displaystyle\int_{\delta_R} \frac{e^{inz}}{z^m}\, dz \longrightarrow 0$, and $\int_{\delta_\epsilon} \frac{e^{inz}}{z^m}\, dz \longrightarrow -\pi i \mathrm{Res}\left(\frac{e^{inz}}{z^m}, 0\right)$.

Note that $\mathrm{Res}\left(\frac{e^{inz}}{z^m}, 0\right) = \lim\limits_{z\to 0} \dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left(z^m \dfrac{e^{inz}}{z^m}\right) = \dfrac{(in)^{m-1}}{(m-1)!}$.

Therefore, $\lim\limits_{R\to\infty}\left(\displaystyle\int_R^R \frac{e^{inx}}{x^m}\, dx\right) = \pi \dfrac{i^m\, n^{m-1}}{(m-1)!}$

and the result of the proposed integral must be a rational multiple of π. □

E-mail address: aplaza@dmat.ulpgc.es