**11281.** Proposed by Mas Alekseyev, University of California-San Diego, La Jolla, CA, and Emeric Deutsch, Polytechnic University, Brooklyn, NY.

Show that the number of permutations $\pi$ of $\{1,\ldots,n\}$ such that $\pi(k)-k$ takes exactly two distinct values is equal to $\sigma(n)-\tau(n)$, where $\sigma(n)$ is the sum of the divisors or $n$ and $\tau(n)$ is the number of divisors.

**Solution:** *(by Ángel Plaza, ULPGC, 35017-Las Palmas G.C., Spain, e-mail: aplaza@dmat.ulpgc.es)*

The solution is based in the following facts:

a) Let $n$ be a prime number. Then each permutation $\pi$ verifying the condition of the problem is a cyclic permutation.

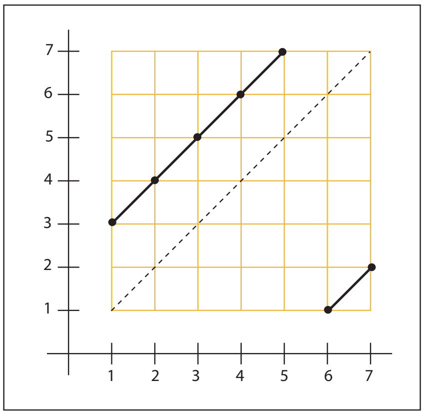


Figure 1: Example of valid permutation for the set $\{1,\ldots,7\}$

Let $n$ be a prime number and let $\pi$ be a permutation of $\{1,\ldots,n\}$ such that $\pi(k)-k$ takes exactly two distinct values for $k\in\{1,\ldots,n\}$. Note that a permutation of $\{1,\ldots,n\}$ is a bijection of the set and therefore $\sum_{1\le k\le n}\pi(k) = \sum_{1\le k\le n}k$. So if there is a $k$ such that $\pi(k)>k$ it should be another $k^*$ with $\pi(k^*)<k^*$. Consequently all the permutations of $\{1,\ldots,n\}$ such that $\pi(k)-k$ take exactly two

distinct values, take one positive value and one negative value and there are not fixed points. These two values are determined only by the image of 1, that is by the value $\pi(1)$, as can be easily seen in Fig. 1 for the set $\{1, \ldots, 7\}$, and the value $\pi(1) = 3$. In fact, these permutation are known as cyclic permutations and can be written explicitly by $\pi(k) = \pi(1) + k - 1 \pmod{k}$. From the example shown in Fig. 1 the other value for $\pi(k) - k$ is straightforwardly deduced.

b) Let $n$ a composite number and $m > 1$ such that $m \mid n$.

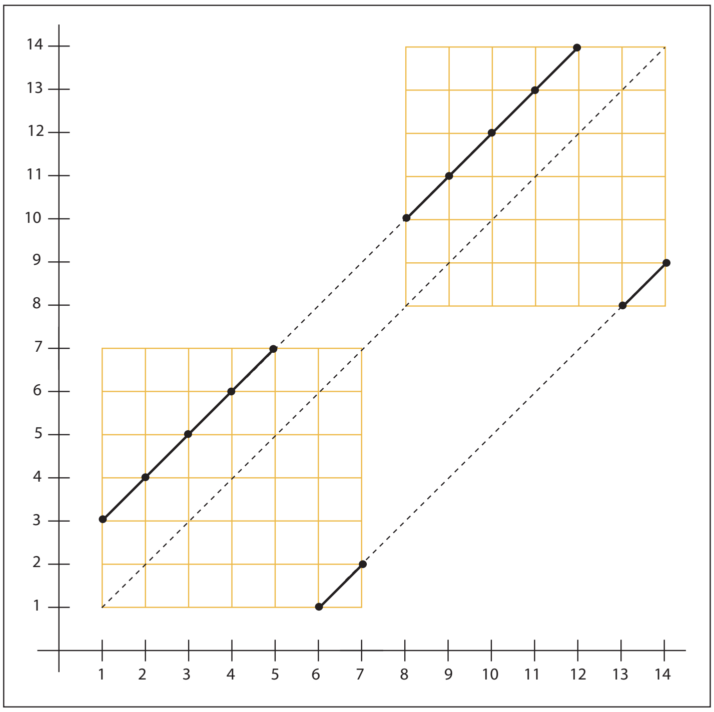


Figure 2: Example of valid permutation for the set $\{1, \ldots, 14\}$

Then $n = m \cdot d$ for some integer $d$. Then the $d$ times product of the same cyclic permutation of the subsets $\{1, \cdot, m\}$, $\{m + 1, \cdot, 2m\}$, $\cdot$ $\{(d-1)m + 1, \cdot, n\}$ is an element verifying the condition of the problem. And in the case of $n$ composite these are the only permutations of the set we are looking for. An example of this situation is shown in Fig. 2 for the set $\{1, \ldots, 7\}$, and the value $\pi(1) = 3$.

It should be noted that for the case of $n$ prime and also for the case $n$ composite there are no other solutions. Fig. 3 shows an non-valid example of a permutation of the set $\{1, \ldots, 12\}$, for the same initial value $\pi(1)$ and in which the set is considered in two subsets: $\{1, \ldots, 7\}$ and $\{8, \ldots, 12\}$. The value $\pi(1)$ determines a valid cyclic permutation of $\{1, \ldots, 7\}$. However, since the remaining values have to be on the dashed lines, permutation $\pi$ can not be completed for the whole set $\{1, \ldots, 12\}$ in the conditions of the problem.

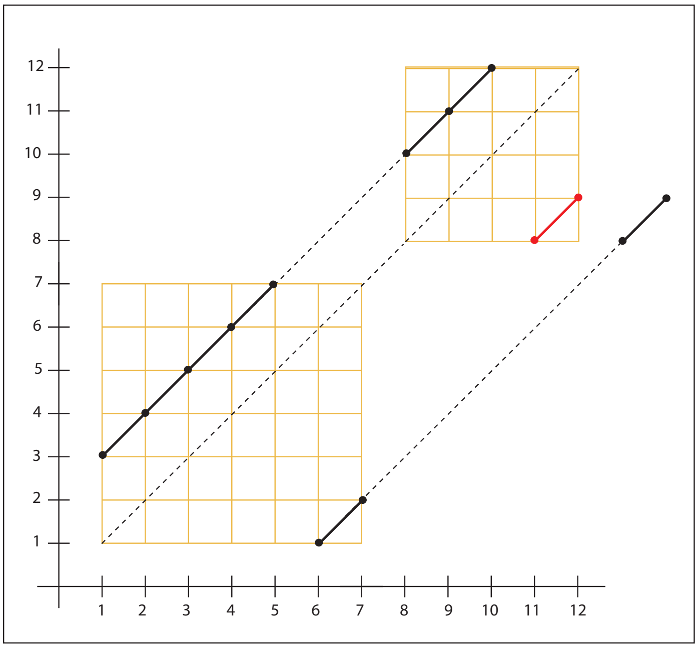


Figure 3: Example of a non-valid permutation for the set $\{1, \ldots, 12\}$

c) The number of cyclic permutations of a set of $m$ different elements is $m - 1$. Note that every (valid) permutation is determined by the value of $\pi(1)$. But $\pi(1) \neq 1$ so there are $m-1$ possible value for $\pi(1)$.

Now, the conclusion of the problem is straightforwardly deduced.

For each divisor $m$ of $n$, with $m > 1$ there are $m-1$ permutations of interest. Then the number of such permutations is:

$$\sum_{m \mid n} m - 1 = \sum_{m \mid n} m - \sum_{m \mid n} 1 = \sigma(n) - \tau(n)$$