Problem 11288 (Proposed by Christopher Hillar, Texas A&M University, College Station, TX and Troels Windfeldt, University of Copenhagen, Copenhagen, Denmark)

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Let $n$ be a positive integer, and let $U = \{1, \ldots, 2n\}$. For a set $S \subseteq U$ and a positive integer $d$, let $h_S^d$ be the sum of all monomials of degree $d$ in the indeterminants $\{X_i : X_i \in S\}$. Let $\tau$ be the set of all $n$-elements subsets of $U$ with the property that for any odd element $k$ of the set, $k+1$ is not a member. For $S$ in $\tau$, let $o(S)$ denote the number of odd elements of $S$. Show that for every positive integer $d$,

$$h_U^d \prod_{i=1}^{n} (X_{2i-1} - X_{2i}) = \sum_{S \in \tau} (-1)^{o(S)} h_{U \setminus S}^{d+n}$$

**Solution:**

If $M$ is a monomial and $P$ a polynomial such that $M$ appears in $P$, we will write $M \in P$.

Let $\mathcal{R}$ denote the set of the monomials in the right-had side (RHS) of previous equation with their respective sign, and let $\mathcal{L}$ de set of the monomials in the left-hand side (LHS) of the same equation, again with their respective sign. Our goal is to show that $\mathcal{R} = \mathcal{L}$ if possible. In fact, we shall show that after cancellation in each side of the equation, both sets are the same.

For $M$ in $\mathcal{R}$, there exist $M_d \in h_U^d$, and $M_n \in \prod_{i=1}^{n} (X_{2i-1} - X_{2i})$ such that $M = M_d \cdot M_n$. Two possibilities are presented now.

(1) If the composition $M = M_d \cdot M_n$ is unique, then monomial $M_n \in \prod_{i=1}^{n} (X_{2i-1} - X_{2i})$ is determined by the elements that we will denote as $\hat{X}_i \in \{X_{2i-1}, X_{2i}\}$ for $i = 1, \ldots, n$. Let $S^*$ be the set of the $n$ indeterminants appearing in $M_n$. That is, $S^* = \{\hat{X}_i : i = 1, \ldots, n\}$.

In addition, if the sign of $M$ is denoted by $sign(M)$, then $sign(M) = (-1)^{e(S^*)}$, where $e(S^*)$ denote the number of even elements of $S^*$.

Notice that $S^*$ verifies that if $X_{2i} \in S^*$, then $X_{2i-1} \notin S^*$. Therefore, if $S = U \setminus S^*$, then $o(S) = e(S^*)$, and if $X_{2i-1} \in S$, then $X_{2i} \notin S$. This means that $S \in \tau$.

Since in this case, the composition $M = M_d \cdot M_n$ is unique, then all the indeterminants of $M_d$ are in $M_n$, since otherwise $M_n$ would not be unique. Therefore, $M \in (-1)^{o(S)} h^{d+n}_{U \setminus S}$. So $M \in \mathcal{L}$.

(2) Let us suppose now that the composition $M = M_d \cdot M_n$ is not unique. This implies that for some pair $\{X_{2i-1}, X_{2i}\}$ both indeterminants appear in $M$. Let $j$ be in $\{1, \ldots, n\}$ such that $X_{2j-1}, X_{2j} \in M$ and let us denote by $M_{n_o}$ the monomial of degree $n$ in $\prod_{i=1}^{n} (X_{2i-1} - X_{2i})$ in which appears $X_{2j-1}$, and let us denote by $M_{n_e}$ the monomial of degree $n$ in $\prod_{i=1}^{n} (X_{2i-1} - X_{2i})$ such that $X_{2j}$ is in there. Note that then $sig(M_{n_o}) = -sig(M_{n_e})$, and therefore, for each $M$ in $\mathcal{R}$ with no unique composition $M = M_d \cdot M_n$, there exists other monomial $M^*$ in $\mathcal{R}$ with opposite sign, so they cancell each other.

This shows that after cancellation in (RHS) $\mathcal{R} \subset \mathcal{L}$ .

It should be shown now that, after cancellation in (LHS), $\mathcal{L} \subset \mathcal{R}$. To this end, consider $S \in \tau$ and let $S^* = U \setminus S$. For each $M \in (-1)^{o(S)} h^{d+n}_{U \setminus S} = (-1)^{e(S^*)} h^{d+n}_{S^*}$, there are two possibilities:

(1) If there are $n$ indeterminants in $M$, then $M \in \mathcal{R}$.

(2) In other case, let us suppose for instance that for the same $j \in \{1, \ldots, n\}$, $X_{2j-1} \notin M$, and $X_{2j} \notin M$. Since either $X_{2j-1}$, or $X_{2j}$ belongs to $S$, there exists $M^*$ in $\mathcal{L}$ such that $sign(M^*) = -sign(M)$, and therefore they cancell each other. $M^*$ is related with $M$ by simple changing $X_{2j-1}$ by $X_{2j}$ in $S$ to get the corresponding subset $S^*$. Explicitely we set $S^* = S \setminus X_{2j} \cup X_{2j-1}$, if $X_{2j} \in S$, and $S^* = S \setminus X_{2j-1} \cup X_{2j}$, if $X_{2j-1} \in S$.

This proves that after cancellation in (LHS), $\mathcal{L} \subset \mathcal{R}$.