A Variant Intermediate Value

11290 [2007, 359]. Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal Highschool, Constanza, Romania. Let f and g be continuous real-valued functions on [0, 1]. Prove that there exists c in (0, 1) such that

$$\int_{x=0}^{1} f(x) \, dx \int_{x=0}^{c} xg(x) \, dx = \int_{x=0}^{1} g(x) \, dx \int_{x=0}^{c} xf(x) \, dx.$$

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PROBLEMS AND SOLUTIONS

Solution by Kenneth F. Andersen, University of Alberta, Edmonton, AB, Canada. Observe first that if h(x) is continuous on [0, 1] and $H(x) = \int_0^x yh(y) dy$, then H(x) is continuous on [0, 1] with $\lim_{x\to 0^+} H(x)/x = 0$, so an integration by parts yields

$$\int_{0}^{1} h(x) dx = \int_{0}^{1} \frac{xh(x)}{x} dx = \frac{H(x)}{x} \Big|_{0}^{1} + \int_{0}^{1} \frac{H(x) dx}{x^{2}}$$
$$= H(1) + \int_{0}^{1} \frac{H(x) dx}{x^{2}} = \lim_{x \to 1^{-}} H(x) + \int_{0}^{1} \frac{H(x) dx}{x^{2}}.$$
 (1)

Now suppose in addition that $\int_0^1 h(x) = 0$. By (1), H(x) cannot be positive for all x in (0, 1), nor can it be negative for all x in (0, 1). Thus by the Intermediate Value Theorem there is a $c_h \in (0, 1)$ such that $H(c_h) = 0$. Now the required result may be deduced: if $\int_0^1 f(x) dx = 0$, then the result holds with $c = c_f$; if $\int_0^1 g(x) dx = 0$, then the result holds with $c = c_h$, where

$$h(x) = \frac{f(x)}{\int_0^1 f(y) \, dy} - \frac{g(x)}{\int_0^1 g(y) \, dy}$$

Editorial comment. (i) The functions f and g need not be continuous—it is sufficient that they be integrable. This was observed by Botsko, Pinelis, Schilling, and Schmuland. (ii) Keselman, Martin, and Pinelis noted that $\int_0^1 xf(x) dx$ and $\int_0^1 xg(x) dx$ can be replaced with $\int_0^1 \phi(x) f(x) dx$ and $\int_0^1 \phi(x)g(x) dx$, where $\phi(x)$ satisfies suitable conditions—roughly speaking, that ϕ is differentiable and strictly monotonic, although the specific conditions vary from one of these solvers to another.

Also solved by U. Abel (Germany), S. Amghibech (Canada), M. W. Botsko & L. Mismas, R. Chapman (U. K.), J. G. Conlon & W. C. Troy, P. P. Dályay (Hungary), J. W. Hagood, E. A. Herman, S. J. Herschkorn, E. J. Ionascu, G. L. Isaacs, G. Keselman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), G. Martin (Canada), J. Metzger & T. Richards, M. D. Meyerson, A. B. Mingarelli & J. M. Pacheco & A. Plaza (Spain), E. Mouroukos (Greece), P. Perfetti (Italy), I. Pinelis, M. A. Prasad (India), K. Schilling, B. Schmuland (Canada), H.-J. Seiffert (Germany), J. Sun, R. Tauraso (Italy), M. Tetiva (Romania), L. Zhou, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.